MATHEMATICAL QUESTIONS

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

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CORRIGENDA.

VOL. XXII.

Pages viii. and 27, *for* No. of Question 4350, *read* No. 4312

VOL. XXVIII,

Page 68, last line, *for* (r^2x^2) *read* (r^2-x^2).
Page 69, line 7 from bottom, *for* (R^2-z^2) *read* $(R^2-z^2)^{\frac{1}{2}}$
Page 86, line 9 from bottom, *for* inssde *read* inside.

VOL. XXIX.

Page 40, line 12 from bottom, *for* No. 133 *read* 134.
Page 59, line 4 from bottom, *after* "values" *insert* "of."
Page 60, line 10, *for* LdL *read* L^3dL.
Page 61, line 13 from bottom, *read* "the respective probabilities of the promiscuous occurrence of the two cases are $\frac{1}{3}$ and $\frac{2}{3}$."
Page 74, line 11, *for* No. 136 *read* 137.
Page 90, line 15, *for* 2, 8, 6 *read* 2, 3, 6.
Page 99, line 12 from bottom, *for* No. 137 *read* 139.
Page 106, line 10, *for* No. 139 *read* 140.

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CONTENTS.

Mathematical Papers, &c.

Solved Questions.

35

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

NOTES ON RANDOM CHORDS. *By the* EDITOR.

Since the publication of Miss BLACKWOOD's Solution of Question 5461, with Mr. WOOLHOUSE's notes thereon (*Reprint*, Vol. XXVIII., pp. 108—110), we have received several communications on the subject of random chords, from which we now select for publication the following remarks:—

1. Colonel CLARKE's views on the subject are expressed as follows:—

A question very similar to 5461 was proposed by Mr. WOOLHOUSE in the *Lady's and Gentleman's Diary* for 1856, in the following terms:—"In a dark room two persons each of them draw a chord at random across a circular slate; what is the chance that they will intersect?" The solutions adopted at the time gave $\frac{1}{3}$ as the answer; therein agreeing with Miss BLACKWOOD's. But there is another solution which gives $\frac{1}{2}$ as the chance; and these are the solutions of these problems:—

(A) "Two random points P_1, P_2 on the circumference of a circle are joined by a straight line, and two other random points Q_1, Q_2 are joined by a straight line; what is the chance that these lines intersect within the circle?" The answer here is, without question, $\frac{1}{3}$. Or, expressed otherwise, thus:—(A) "Through each of two random points P, Q on the circumference of a circle, a chord is drawn in any direction at random, what is the chance that these chords intersect within the circle?"

(B) "Two lines are drawn at random across a circle, what is the chance that they intersect within the circle?" The answer here is, without question, $\frac{1}{2}$.

In (A) the distribution of lines or chords is that of lines drawn through random points on the circumference of a circle. But such lines are not random lines; since for a given direction they lie closer together towards the circumference, and are further apart at the centre; they are not *random* lines, or they could not have this particular law of density. In Question 5461, the expression is "all possible chords being supposed equally probable." Here we must suppose an even distribution of chords. In problem (A) the chords are distributed evenly as to direction, but *not* as to distance from the centre; and we might evidently have even distribution as to direction combined with any arbitrary law of distribution as

to distance from the centre. But the only admissible law of distribution as to distance from the centre is the law of uniformity; they must not be closer towards the circumference than they are at the centre. If we suppose the chords drawn so that they are distributed evenly as to direction, and also evenly as to distance from the centre, the probability of two chords intersecting is certainly $\frac{1}{2}$. This is the solution of problem (B), and it is in accordance with Professor CROFTON's system or definition of random lines. On this principle, I had given a Solution of Question 2109 (*Reprint*, Vol. XI., p. 94), before Professor CROFTON had given his theory of random lines, which Mr. WOOLHOUSE characterizes as peculiar. But Mr. WOOLHOUSE himself had adopted this system, and extended it to space, in solving the following problem of his own in the *Diary* for 1860:—"A cube is thrown into the air at random, and a shot fired through it, what is the chance the shot passes through opposite faces of the cube?" To get the result, Mr. WOOLHOUSE determines for any one definite direction, (1) the number of shot (or parallel lines) which, evenly distributed, would pass through the cube supposed fixed: (2) the number—being part of the preceding—which pass through two opposite faces; (3) embracing all directions evenly, he determines the total number of lines which intersect the cube; and (4) the total number which pass through two opposite faces; the ratio of these last numbers being the required result. In the same number of the *Diary*, another correspondent solves the question by supposing the shot to enter one particular face, and from that entrance point—the cube being fixed—to take any direction at random. This solution Mr. WOOLHOUSE shews to be erroneous. We have in this case random lines through random points in a face of the cube taken in place of random lines. And on the same principle I object to the solution that gives $\frac{1}{3}$ as the answer to Question 5461, because in that solution lines drawn through random points in a circle are taken for random lines. But we must not confound random lines through random points on a limited space, with random lines wholly unrestricted.

2. On sending the foregoing remarks to Professor CROFTON, we received from him the following note thereon:—

I quite agree with Col. CLARKE's remarks on Quest. 5461. Of course the results will depend on the manner in which we suppose "a random chord" of the area to be drawn. The results obtained by Miss BLACKWOOD and others are quite correct on the supposition that two points are taken on the perimeter of the circle (or any other convex figure) at random, and joined. I would observe, however, that it would be just as natural to take any two points at random within the circle and join them; but the result would be quite different; the problem indeed would, in this form, be a very difficult one, and would be an interesting exercise for our contributors. My own idea of the most natural sense of the term is a straight line drawn at random in the plane, and meeting the area—that is, the plane is supposed covered by an infinite number of random lines, and those which cross the area alone are considered; one of them being taken at random: exactly as a random point might be supposed to be selected from those, out of an infinite number scattered over the plane, which happen to fall inside the boundary in question. Thus we might imagine an infinity of points to fall like drops of rain all over the plane, and only consider those that fall on the figure: and, in the former case, an infinity of *lines* thrown down at random on the plane in the same way. Of course it comes to the same thing to suppose a *single point* thrown at random an infinite number of times on the plane; or a single straight line; and its position marked out every time it falls on the boundary. This coincides with M. BERTRAND's view,

which is expressed as follows on p. 487 of his *Calcul Intégral*:—"Concevons sur un plan une série de lignes parallèles équidistantes, et supposons que l'on projette au hasard sur ce plan un disque de forme convexe de dimensions assez petites pour ne pouvoir pas rencontrer deux lignes à la fois . . . . et qu'on recommence l'épreuve jusqu'à ce qu'il y ait intersection. La corde détachée par celle des lignes qui est rencontrée sera pour nous la première des cordes choisies au hasard; la seconde s'obtiendra par une autre épreuve, renouvelée, s'il est nécessaire."

3. Mr. G. S. Carr, referring more especially to the form that Mr. Woolhouse gave to the question, writes as follows:—

If two persons enter a dark room and draw independently a line with a ruler upon a circular slate, it appears that what would happen would be this:—Each person would approach the circle indifferently from any point of the compass, and, holding the ruler before him in the usual manner, would place it upon the circle at any indifferent distance from himself. Consequently, the two random lines so drawn would fall under Professor Crofton's definition of random lines, and the probability of their intersecting would certainly be $\frac{1}{3}$. It would seem that a line drawn through one or two points, even though those points be taken at random on a given line or area, is to be considered "at random" only in a restricted sense. But the definition founded upon the two elements of angular direction and translation alone, is absolute and covers all cases.

4. Miss Blackwood, after reading a "proof" of the foregoing part of these Notes, calls the Editor's attention to the fact that Mr. Woolhouse has "pronounced her Solution of Question 5461 to be clear and satisfactory, and needing no confirmation," and then, remarking that she has thus "drawn upon herself the heavy artillery of Colonel Clarke and his allies," replies thereto as follows:—

It is positively refreshing to find Mathematicians now and then throwing aside their ponderous mathematical armour, and indulging in a pleasant set-to on the neutral ground of metaphysics. At the same time, it seems to me that the practised logician has here the same unfair advantage over the mathematician that Fitz-James had over the chivalrous but ill-fated Roderick Dhu at Coilantogle Ford, when the latter threw down his "targe," and trusted to his sword alone.

The question at issue is the purely metaphysical one, what should be meant by the phrase "a *random chord*." The logician (Fitz-James) says to the mathematician (Roderick Dhu), "Let us strike out the word *random* for a moment, and consider what is the ordinary accepted definition of the simple word *chord*." Referring to a mathematical text-book, they find and accept the definition, "A straight line joining two points in the circumference of a circle is called a *chord*." "*What* two points in the circumference?" asks the logician. "*Any* two points you like to take," answers the mathematician; and, so saying, he exposes himself to the decisive thrust, "Then, when the points are *random points*, the chord must, as a necessary consequence, be a *random chord*."

5. Commenting on a "proof" sent to him of Nos. (1), (2), (3) of the foregoing notes, Mr. Woolhouse writes as follows:—

In these *Notes on Random Chords* no allusion whatever is made to the principal matter, viz., the errors pointed out, upon which an exercise of judgment might have been of some service. The notes, indeed, are merely responses to a passing observation, which must be under-

stood as having exclusive reference to the subject of "random chords drawn in a circle," that Professor CROFTON's theory when applied to such chords is peculiar, and that results derived from it, should be regarded as special. My meaning is more fully stated in the *Note on Random Lines* (*Reprint*, Vol. X., p. 33). It would seem, however, that Colonel CLARKE had written his remarks without giving much, if any, consideration to that Note, although it is expressly referred to by me. The only divergence of opinion does not relate to any process of investigation, but simply to the definition given to "random chords" at the outset, which to a certain extent may be considered as optional, even at the risk of being peculiar, provided the same be clearly explained. I have, however, before stated, and am prepared to maintain, that the mathematical condition to which random lines are subject, when not specially defined, is not arbitrary, but should in every case be elicited from the practical nature of the problem, as suggestive of the particular mode in which the lines may happen to be generated. The cube and circular-slate questions are both accurately solved on this undoubtedly rational and correct principle. With respect to the *modus operandi* of drawing the chord across the circular slate, I object to the arbitrary description given by Mr. CARR. I hold that each person would, in the first instance, instinctively ascertain a locality in the periphery from whence to draw the chord, and that the chord so drawn would therefore be a random line proceeding from a random point in the circumference, and would not fall under Professor CROFTON's general definition of random lines. In short, I consider Professor CROFTON's definition as not adapted to ordinary notions of the generation of a random chord; but at the same time there can be no doubt whatever, that it is the best definition of a "random line on an indefinite plane." Adapting the alternative enunciation to the case before us, I am therefore of opinion (1) that the probability of intersection of "two random chords drawn in a circle," is $\frac{1}{3}$; and (2) that, if "two random lines meet a circle" the probability that they shall intersect within the circle is $\frac{1}{2}$. If any other mathematicians perceive sufficient reason to adopt the latter as "random chords," it is no business of mine.

4870. (By Professor CAYLEY, F.R.S.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find on the first a point A′, on the second a point B′, and on the third a point C′, such that the intersections of the lines

A′B′ and AC, A′C′ and AB, lie on the first conic;
B′C′ and BA, B′A′ and BC, lie on the second conic;
C′A′ and CB, C′B′ and CA, lie on the third conic.

Solution by J. HAMMOND, M.A.; Prof. EVANS, M.A.; *and others.*

Taking ABC as the triangle of reference, the three conics are

$$u \equiv by^2 + cz^2 + fyz + gzx + hxy = 0, \quad v \equiv a'x^2 + c'z^2 + f'yz + g'zx + h'xy = 0,$$
$$w \equiv a''x^2 + b''y^2 + f''yz + g''zx + h''xy = 0,$$

subject to the condition $w \equiv u + \lambda v$, since they pass through the same four points. Now B′C′ passes through the intersections of BA and v and of CA and w, that is, B′C′ passes through the points

$$(a'x + h'y = 0,\ \ z = 0), \qquad (a''x + g''z = 0,\ \ y = 0),$$

its equation is therefore $\mathrm{P} \equiv a'a''x + a''h'y + a'g''z = 0.$

Similarly, the equations of C′A′ and A′B′ are

$$\mathrm{Q} \equiv b''hx + b''by + bf''z = 0, \qquad \mathrm{R} \equiv c'gx + cf'y + cc'z = 0.$$

Now $$\frac{\mathrm{Q}y}{b''} + \frac{\mathrm{R}z}{c'} \equiv by^2 + cz^2 + gzx + hxy + yz\left(\frac{bf''}{b''} + \frac{cf'}{c'}\right)$$

$$\equiv u + yz\left(\frac{bf''}{b''} + \frac{cf'}{c'} - f\right).$$

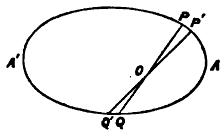
Also, since $w \equiv u + \lambda v$, we have $f'' = f + \lambda f'$, $b'' = b$, $c + \lambda c' = 0$;

thus $$\frac{\mathrm{Q}y}{b''} + \frac{\mathrm{R}z}{c'} \equiv u, \quad \&c.$$

5474. (By the Rev. A. F. Torry, M.A.)—Find what normal divides an ellipse most unequally.

Solution by H. Pollexfen, B.A.; J. L. McKenzie, B.A.; *and others.*

Let POQ be the required normal, P′OQ′ the consecutive normal. Then the difference between the areas PAQ and PA′Q, being a maximum, is for the moment constant; and the triangles POP′, QOQ′ are therefore equal. Hence POQ is bisected in O; or the chord PQ is the diameter of the circle of curvature at P. But the tangent at any point of an ellipse, and the common chord of the ellipse and its circle of curvature, make equal angles with the axis. Therefore, in the present case, the tangent and normal at P make angles of 45° with the axis. The coordinates of P are



$$x = a^2(a^2 + b^2)^{-\frac{1}{2}}, \quad y = b^2(a^2 + b^2)^{-\frac{1}{2}}.$$

5428. (By Professor Elliott, M.A.)—Prove (1) that the highest point on the wheel of a carriage rolling on a horizontal plane moves twice as fast as each of two points in the rim whose distance from the ground is half the radius of the wheel; and (2) find the rate at which the carriage is travelling when the dirt thrown from the rim of the wheel to the greatest height attains a given level, explaining the two roots of the resulting equation.

Solution by J. J. WALKER, M.A.; Prof. EVANS, M.A.; *and others.*

The first part of the question is obvious, since the velocities of all points on the wheel are proportional to the chords drawn from the point of contact with the ground to those points.

For the second part it is readily found that, V being the velocity of the entre and $V^2 = 2gh$, the height on the wheel from the ground of the particle of mud which reaches the greatest altitude is $r + \frac{r^2}{2h}$, which being necessarily not greater than $2r$, $2h$ must be not less than r. And supposing a to be the greatest height attained, $2h$ is determined by the equation

$$4h^2 - 4(a-r)h + r^2 = 0,$$

so that one of the two values of $2h$ is greater and the other less than r. The latter value is therefore excluded by the nature of the Question.

5437. (By CHRISTINE LADD.)—If I_1, I_2, I_3 be the points of contact of the inscribed circle with the sides of a triangle ABC; O_1, O_2, O_3 the centres of the escribed circles; r_i, r_e the radii of the circles inscribed in the triangles $I_1I_2I_3$, $O_1O_2O_3$; and α, β, γ the distances O_2O_3, O_3O_1, O_1O_2;

prove that $$\frac{r_i}{r} = \frac{r_e}{2R} = \frac{a+b+c}{\alpha+\beta+\gamma}.$$

Solution by the EDITOR.

Putting a_1, b_1, c_1, s_1 for the sides and semiperimeter of the triangle $I_1I_2I_3$, we have $a_1 = 2r\cos\frac{1}{2}A$, $b_1 = 2r\cos\frac{1}{2}B$, $c_1 = 2r\cos\frac{1}{2}C$; and as r_i, r are the respective radii of the inscribed and circumscribing circles of this triangle, we have, by a well-known form,

$$4rr_is_1 = a_1b_1c_1, \quad \text{whence } \frac{r_i}{r} = \frac{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C} = \frac{a+b+c}{\alpha+\beta+\gamma},$$

the last form following from Vol. II. of Dr. BOOTH's *New Geometrical Methods*, Secs. 187, 216, where it is proved that

$$4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = s, \quad 4R(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C) = \alpha+\beta+\gamma.$$

Again, the angles of the triangles $I_1I_2I_3$, $O_1O_2O_3$ are

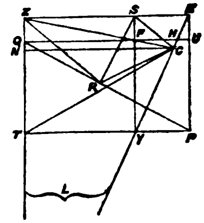
$$\tfrac{1}{2}(B+C), \quad \tfrac{1}{2}(C+A), \quad \tfrac{1}{2}(A+B);$$

hence these triangles are similar; and as the triangle ABC is the orthocentric triangle of the triangle $O_1O_2O_3$, 2R is the radius of the circle that circumscribes the triangle $O_1O_2O_3$;

therefore $$\frac{r_i}{r} = \frac{r_e}{2R}.$$

5395. (By C. Smith, M.A.)—**If P, Q are points on two confocal conics, such that the tangents at these points are at right angles, show that the line PQ envelops a third confocal.**

Solution by R. F. Davis, B.A.; S. Johnston, M.A.; *and others.*



Let TP, TQ be two tangents one to each of two confocal conics; S a common focus, and C the common centre. Draw the perpendiculars SY, SZ; join CY, CZ, meeting SZ, SY respectively in E, F. Then it is easily seen that EP, FQ are parallel to SY, SZ. Complete the rectangle QTPU, and let CY meet QU in H and TQ in L. Draw the perpendiculars SR, CN, and join CR, ZR.

Then, since SYTZ is a rectangle, and C any point in its plane, $SC^2 + CT^2 = CY^2 + CZ^2$; hence CT is constant [and $= a^2+b^2-\lambda^2$, if a^2, b^2 and $a^2-\lambda^2$, $b^2-\lambda^2$ be the squares of the semiaxes]. Moreover, since $CL : CY = CZ : CF = CE : CH$,

therefore $CL : CE = CY : CH$, or $CL : EL = CY : YH = TN : TQ$. But $CL : EL = CN : EZ$ or QU; hence $TN : NQ = CN : QU$, and CT passes through U. The point C may therefore be regarded in somewhat of the light of a "centre of similitude" of the three rectangles SP, SQ, PQ, one diagonal of each passing through it.

Next, since a circle can be circumscribed about SZQRF, we have

$$\angle SRZ = SFZ = CZT, \quad \angle RSZ = \tfrac{1}{2}\pi - YSR = \tfrac{1}{2}\pi - QPT = CTZ;$$

therefore the triangles SRZ, CZT are similar, and

$$CT : SZ \text{ or } TY = CZ : ZR.$$

Finally, $\angle CZR = FQR = CTY$, and the triangles CTY, CZR are similar. Hence $CR \, . \, CT = CY \, . \, CZ$, or CR is constant [and $= a^2 (a^2-\lambda^2)(a^2+b^2-\lambda^2)^{-1}$].

4253. (By H. S. Monck, M.A.)—**A series of Pythagorean triangles with the difference between the hypothenuse and one side equal to *n*, can always be obtained by beginning with the triangle 3*n*, 5*n*, 4*n*, and taking the upper figure as negative in each odd term of the series given in Quest. 4102. Find in what cases a *distinct* series with the same difference can be obtained.**

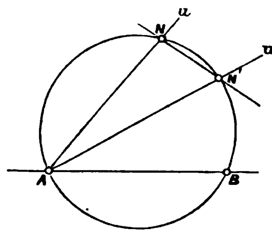
Solution by the Proposer.

A distinct series is possible if n be a square number or any multiple of a square number (other than unity). For the three sides of a Pythagorean triangle are always of the form $m(2ab)$, $m(a^2+b^2)$, and $m(a^2-b^2)$; and hence the difference between the hypotenuse and one side is either $m(a-b)^2$ or $2m \, . \, b^2$, either of which are multiples of a square number. Conversely, if $n = mx^2$, we can form a series by putting $x = a-b$, and continuing backwards and forwards; or, if m be an even number, we can put $x = b$, and taking any larger number for a, continue the series backwards and forwards as before.

5462. (By Professor ENGLAND, M.A.) — A variable circle passes through two given points, and through one of these pass two given lines; find the envelope of the chord that joins the other points where the circle intersects these lines.

Solution by JACCOBINI VINCENZO, *and* SIMONELLI RUGGERO.

Sieno A e B i due punti ed u, u' le due rette date. Si prenda sulla u un punto qualunque N; per questo e per A e B si faccia passare un cerchio; esso determinerà sulla u' un unico punto N′.



Se ora sulla u' si prende il punto N′; il cerchio che passa per N′, A, B andrà ad intersecare la u nell unico punto N.

Quindi ad un punto della u, corrisponde un unico punto nella u', e viceversa. Le due rette u, u' sono per tal fatto proiettive e l'inviluppo richiesto è quindi una *conica*.

Osservando poi che fra gl'infiniti cerchi che passano per A e B vi è quello che si scinde nella congiungente AB e nella retta all' infinito del piano, esso taglierà le due punteggiate u ed u' nei punti all' infinito. Per cui, appartenendo la retta all' ∞ all' inviluppo, la conica è una *parabola*.

5472. (By CECIL SHARPE.)—Show how to draw a straight line terminated by the circumferences of two of three given circles, and bisected by the third circumference.

Solution by H. POLLEXFEN, B.A.; C. BICKERDIKE; *and others*.

Let O be the centre of the circle (1). Take any point P on the circumference of (2); join OP and bisect it in C. With C as centre and a radius equal to half the radius of (1) draw a circle, cutting (3) in Q and R. Then it is evident that P is a centre of similitude of the circles whose centres are O and C, and that any line drawn from P to the circle O is bisected by C. Hence PQ and PR are two of the required lines.

5520. (By WILLIAM R. ROBERTS, M.A.) — Shew that the six inflexional tangents to a unicursal quartic all touch the same conic.

Solution by J. C. MALET, M.A.; H. T. GERRANS, M.A.; *and others*.

Projecting two of the double points to the circular points at infinity, and inverting from the remaining double point, the question is reduced to

the following :—" the centres of the six circles that can be drawn through a fixed point to osculate a given conic, lie on a conic," which may be proved as follows :—Take the fixed point as origin, and let the equations of the given conic and of any circle through the origin be

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, \quad x^2 + y^2 - 2\alpha x - 2\beta y = 0.$$

Now, if two conics U and V osculate, the discriminant of $U + kV$, viz., $\Delta + \Theta k + \Theta' k^2 + \Delta' k^3$ must be a perfect cube; hence we have $3\Delta\Theta' = \Theta^2$ as one condition; in the present case, this gives us

$$3\Delta\,(b\alpha^2 + a\beta^2 - 2h\alpha\beta - 2g\alpha - 2f\beta - c) \\ + \left\{2\alpha\,(bg - fh) + 2\beta\,(af - gh) + (a + b)\,c - f^2 - g^2\right\}^2 = 0,$$

which proves the theorem.

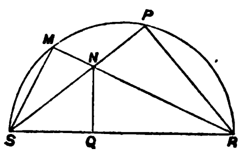
5487. (By Byomakesa Chakravarti.)—If SMPR be a semicircle on SR, and the chords SP, RM cut each other in N; prove that

$$\overline{SR}^2 = SN \,.\, SP + RN \,.\, RM.$$

Solution by C. Bickerdike; C. Vincenzo; S. Ruggero; *and others.*

Draw NQ perpendicular to SR; then, since circles will clearly go round the quadrilaterals SMNQ, RPNQ, we have

$$SP \,.\, SN + RM \,.\, RN \\ = SR \,.\, SQ + SR \,.\, RQ = \overline{SR}^2.$$



5067. (By S. Tebay, B.A.) — Let $x_1 + x_2 + \dots\dots + x_n = 1$, where $x_1 > x_2 > \dots\dots > x_n$; find the mean value of x_r^2.

Solution by the Proposer.

Let $t_r = x_1 + x_2 + \dots\dots + x_r$; then, by Question 4616, we have to find

$$\frac{1}{r-1!\; r-2!} \iiint \dots (t_r - rx\;)^{r-2} x_r^2\, dx_r\, dx_{r+1}\, dx_{r+2} \dots .$$

The first integral from $x_r = x_{r+1}$ to $x_r = \frac{1}{r}\,t_r$ is

$$\frac{1}{r!\; r-1!}\left\{t_{r+1} - (r+1)\,x_{r+1}\right\}^{r-1} x_{r+1}^2 + \frac{2}{r} \cdot \frac{1}{(r\,!)^2}\left\{t_{r+1} - (r+1)\,x_{r+1}\right\}^r x_{r+1} \\ + \frac{2}{r^2} \cdot \frac{r+1}{(r+1\,!)^2}\left\{t_{r+1} - (r+1)\,x_{r+1}\right\}^{r+1}.$$

The third term gives, by successive integration, $\frac{2}{r^2}\cdot\frac{n+1}{(n+1!)^2}$. Multiply the second term by dx_{r+1}, and integrate from $x_{r+1}=x_{r+2}$ to $x_{r+1}=\frac{t_{r+1}}{r+1}$, and we have

$$\frac{2}{r}\cdot\frac{1}{(r+1\,!)^2}\left\{t_{r+2}-(r+2)x_{r+2}\right\}^{r+1}x_{r+2}$$

$$+\frac{2}{r(r+1)}\cdot\frac{r+2}{(r+2\,!)^2}\left\{t_{r+2}-(r+2)x_{r+2}\right\}.$$

The second part gives by successive integration, $\frac{2}{r(r+1)}\cdot\frac{n+1}{(n+1\,!)^2}$. Similarly the first part leads to a term $\frac{2}{r(r+2)}\cdot\frac{n+1}{(n+1\,!)^2}$. And so on. Collecting these, we have

$$\frac{2}{r}\cdot\frac{n+1}{(n+1\,!)^2}\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{r}\right);$$

the mean value of which is

$$\frac{2}{n(n+1)}\cdot\frac{1}{r}\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{r}\right)=\frac{2}{n(n+1)}\,\mathrm{R}_r,\ \text{suppose.}$$

Similarly $$\frac{1}{r!\;r-1\,!}\int\left\{t_{r+1}-(r+1)x_{r+1}\right\}^{r-1}x_{r+1}^2\,dx_{r+1}$$

gives $\frac{2}{n(n+1)}\,\mathrm{R}_{r+1}$; and so on. Thus the mean value required is

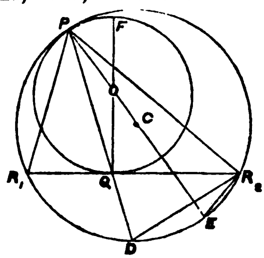
$$\frac{2}{n(n+1)}\cdot\sum_r^n(\mathrm{R}_r).$$

5380. (By W. Gallatly, B.A.)—If a circle A touches internally another circle B at P, and a tangent to A at the point Q intersect B in R_1, R_2, prove that $\angle\,\mathrm{R_1PQ}=\angle\,\mathrm{R_2PQ}$.

[This theorem reciprocates into the following:—S is the common focus of two conics having contact of the second order at P_1; if from a point Q on the outer conic two tangents QR_1, QR_2 be drawn to the inner conic intersecting the common tangent at P in the points R_1, R_2, and if the tangent at Q to the outer conic intersect the tangent at P in T; then TR_1, TR_2 subtend equal angles at the focus.]

Solution by J. S. Jenkins; J. O. Jelly, B.A.; *and others.*

Let O, C be the centres of the circles (A), (B) respectively; then, drawing lines as in the figure, we have



$$\angle\,\mathrm{R_1QF}=\mathrm{ER_2P}=\text{a right angle},$$

and $\angle\,\mathrm{OQP}=\mathrm{OPQ}=\mathrm{DR_2E}$,

therefore $\angle\,\mathrm{R_1QP}=\mathrm{DR_2P}$;

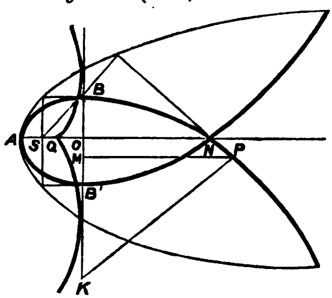
also $\angle\,\mathrm{PR_1Q}=\angle\,\mathrm{PDR_2}$,

therefore also $\angle\,\mathrm{R_1PQ}=\angle\,\mathrm{R_2PQ}$.

5331. (By Professor Wolstenholme, M.A.)—Prove that (1) the evolute of the first negative focal pedal of the parabola $y^2 = c\,(\frac{3}{4}c - x)$, (where $c = 4a =$ the parameter,) is the curve $27\,(y^2 - 8cx - c^2)^2 = 8cx\,(8x + 9c)^2$; (2) the equation of the pedal itself is $27ay^2 = (3a - x)\,(x + 6a)^2$; (3) the normal of the pedal exceeds the ordinate by a fixed length; (4) the arc measured from the vertex to any point is equal to the intercept of the normal on the axis of y; and (5), if a heavy uniform chain be tied tightly round a curve, such that the pressure per unit is equal to the weight of a unit of length of the chain, this curve must be the first negative focal pedal of a parabola.

Solution by the Proposer.

The first negative pedal of the parabola $y^2 = 4a\,(x + a)$ is determined by the equations



$$y \sin\theta - x\cos\theta = \frac{2a}{1+\cos\theta} \;.....(1),$$

$$y\cos\theta + x\sin\theta = \frac{2a\sin\theta}{(1+\cos\theta)^2}\;...(2),$$

(1) being the tangent, and therefore (2) the normal. Hence its evolute is the envelope of (2), and is given by (2) and

$$-y\sin\theta + x\cos\theta = \frac{2a\cos\theta}{(1+\cos\theta)^2} + 4a\,\frac{1-\cos\theta}{(1+\cos\theta)^2},$$

whence $\displaystyle y = -\frac{4a\sin\theta\,(1-\cos\theta)}{(1+\cos\theta)^2}$, or $\displaystyle \left(\frac{y}{4a}\right)^2 = \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^3$,

and $$x = \frac{2a}{(1+\cos\theta)^2}\,(\sin^2\theta + \cos^2\theta + 2\cos\theta - 2\cos^2\theta)$$

$$= 2a - \frac{6a\cos^2\theta}{(1+\cos\theta)^2}, \quad\text{or}\quad \left(\frac{2a-x}{6a}\right) = \left(\frac{\cos\theta}{1+\cos\theta}\right)^2,$$

whence the equation of the evolute will be $\displaystyle \left(\frac{y}{4a}\right)^{\frac{2}{3}} + 2\left(\frac{2a-x}{6a}\right)^{\frac{1}{2}} = 1,$

or, changing the origin from S to O, and reversing the direction of x,

$$\left(\frac{y}{4a}\right)^{\frac{2}{3}} + 2\left(\frac{x}{6a}\right)^{\frac{1}{2}} = 1.$$

This gives $$\left(\frac{y}{4a}\right)^2 = 1 - 6\left(\frac{x}{6a}\right)^{\frac{1}{2}} + 12\,\frac{x}{6a} - 8\left(\frac{x}{6a}\right)^{\frac{3}{2}},$$

or $$\left(\frac{y}{4a}\right)^2 - \frac{2x}{a} - 1 = -\left(\frac{6x}{a}\right)^{\frac{1}{2}}\left(1 + \frac{2x}{9a}\right),$$

or, if $c = 4a$, $$27\,(y^2 - 8cx - c^2)^2 = 8cx\,(8x + 9c)^2.$$

With this equation of the evolute, that of the parabola would be

$$y^2 = c\,(\tfrac{3}{4}c - x),$$

For the pedal itself, we have, with the first origin,

$$x = 2a\frac{1-\cos\theta}{1+\cos\theta} - \frac{\cos\theta}{1+\cos\theta} = 2a\frac{1-2\cos\theta}{1+\cos\theta}, \quad y = \frac{2a\sin\theta}{(1+\cos\theta)^2}(1+2\cos\theta);$$

whence $$\frac{y^2}{4a^2} = \frac{1-\cos\theta}{1+\cos\theta}\left(\frac{1+2\cos\theta}{1+\cos\theta}\right)^2 = \frac{x+a}{3a}\left(\frac{8a-x}{6a}\right)^2,$$

or, moving the origin from S to O, and reversing the direction of x,

$$27ay^2 = (3a-x)(x+6a)^2.$$

Also $\dfrac{dx}{d\theta} = \dfrac{6a\sin\theta}{(1+\cos\theta)^2}$, and $\dfrac{dx}{ds} = \sin\theta$, whence $\dfrac{ds}{d\theta} = \dfrac{6a}{(1+\cos\theta)^2}$,

the intrinsic equation, giving $s = 2a(3\tan\frac{1}{2}\theta + \tan^3\frac{1}{2}\theta)$.

The curve has a crunode at the distance $9h$ from the vertex, the tangents there being inclined at 30° to the axis.

When $\theta = \frac{1}{2}\pi$, $x = 0$, $y = 2a$, giving the point B; when $\theta = \pm\frac{2}{3}\pi$, $y = 0$, the node N; and if we take $\theta > \frac{2}{3}\pi < \pi$, giving y negative as at P, and draw PK the normal at P,

$$\text{PM} = -\frac{6a\cos\theta}{1+\cos\theta}, \quad \therefore \quad \text{PK} = \frac{6a}{1+\cos\theta}, \quad \text{and} \quad \text{PK}-\text{PM} = 6a,$$

or the normal exceeds the ordinate by a constant length.

$$\begin{aligned}\text{Also OK} = \text{OM}+\text{MK} &= -\frac{2a\sin\theta(1+2\cos\theta)}{(1+\cos\theta)^2} - \tan\theta\,.\,\frac{-6a\cos\theta}{1+\cos\theta}\\ &= -\frac{2a\sin\theta}{(1+\cos\theta)^2}(1+2\cos\theta-3-3\cos\theta)\\ &= \frac{2a\sin\theta}{(1+\cos\theta)^2}(2+\cos\theta) = 2a\tan\tfrac{1}{2}\theta\left(1+\frac{1}{2\cos^2\frac{1}{2}\theta}\right)\\ &= a\tan\tfrac{1}{2}\theta(2+1+\tan^2\tfrac{1}{2}\theta) = a(3\tan\tfrac{1}{2}\theta+\tan^3\tfrac{1}{2}\theta) = \text{arc ABP}.\end{aligned}$$

Another obvious property of the curve is, radius of curvature varies as the square of the normal PK.

In the curve in which, if a uniform chain be wrapped round it, the pressure per unit is the weight of a unit, we shall have, if ϕ be the angle which the tangent has turned through from the lowest point,

$$\frac{d\text{T}}{ds} = \kappa g\sin\phi, \quad \text{T}\frac{d\phi}{ds} = \text{R}+\kappa g\cos\phi = \kappa g(1+\cos\phi),$$

whence $$\frac{d\text{T}}{\text{T}d\phi} = \frac{\sin\phi}{1+\cos\phi}, \quad \text{or} \quad \text{T} = \frac{\kappa gh}{1+\cos\phi},$$

and $$\frac{ds}{d\phi} = \frac{\text{T}}{\kappa g(1+\cos\phi)} = \frac{h}{(1+\cos\phi)^2},$$

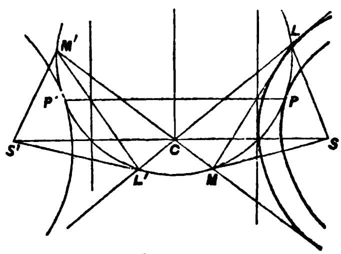
the same equation as we have found for the negative pedal.

The axis of the parabola must be vertical and the vertex downwards, and the string must be so wrapped on that the tension at the lowest point $= \kappa g\,.\,\frac{1}{2}h = \kappa g\,.\,3a =$ weight of a length $3a$ of the chain.

We may suppose the loop of the curve to be made into a separate curve NBAB′N, and the string wrapped round this, the tension at the highest point being four times that at the lowest. [In *Reprint*, Vol. XVI., p. 78, the equation of the pedal, with S as origin, is obtained in the forms $(x+4a)^3 = 27a(x^2+y^2)$, $r = a\sec^3\frac{1}{3}\theta$; and in Vol. XVII., p. 17, the curve is drawn and some of its properties investigated.]

5212. (By Professor WOLSTENHOLME, M.A.)—A circle is drawn touching both branches of a fixed hyperbola in P, P′, and meeting the asymptotes in L, L′, M, M′: prove that (1) LL′ = MM′ = major axis; (2) the tangents at L, M meet in one focus, and those at L′, M′ in the other, and the angle between either pair is constant, supplementary to the angle between the asymptotes; (3) the directrices bisect LM, L′M′; (4) PP′ bisects LL′, MM′, LM, L′M′; (5) the tangents at L, L′ intersect on a rectangular hyperbola passing through the foci and having one of its asymptotes coincident with MM′ (because $\angle$ CSL + $\angle$ CS′L′ = angle between the asymptotes); (6) LM, L′M′ touch parabolas having their foci at the foci of the hyperbola, and the tangents at their vertices the directrices of the hyperbola.

Solution by Professor ARMENANTE; A. MARTIN, M.A.; *and others.*



Let the equation of the fixed hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(1); then a circle drawn touching both branches of the hyperbola will have its centre on the imaginary axis of (1), and its equation will be

$$x^2 + (y-\beta)^2 = r^2 \;\ldots\ldots\; (2).$$

Eliminating x^2 from (1), (2), and putting $a^2+b^2=c^2$, the ordinates of the points of intersection of (1), (2) are

$$y = \frac{\beta \pm \left(\beta^2 - \dfrac{c^2}{b^2}(a^2+\beta^2-r^2)\right)^{\frac{1}{2}}}{\dfrac{c^2}{b^2}};$$

hence the circle will touch the hyperbola if $\beta = \pm \dfrac{c}{a}(r^2-a^2)^{\frac{1}{2}}$, and the coordinates of points (P, P′) of contact will be

$$x = \frac{\pm(a^4+b^2r^2)^{\frac{1}{2}}}{c}, \quad y = \frac{\pm b^2(r^2-a^2)^{\frac{1}{2}}}{ac}.$$

The equation of the circle touching both branches of the hyperbola is, therefore,

$$x^2 + \left(y \mp \frac{c}{a}(r^2-a^2)^{\frac{1}{2}}\right)^2 = r^2 \;\ldots\ldots\ldots\ldots\; (3).$$

Eliminating x^2 from (3), and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ .... (4),

(4) being the equation of the asymptotes of hyperbola, we have, for the coordinates of their points of intersection,

$$\text{L}\left[x_1 = \frac{b(r^2-a^2)^{\frac{1}{2}} - a^2}{c}; \quad y_1 = \frac{-b^2(r^2-a^2)^{\frac{1}{2}} + a^2b}{ac}\right],$$

$$\text{L}'\left[x_2 = \frac{b(r^2-a^2)^{\frac{1}{2}} + a^2}{c}, \quad y_2 = \frac{-b^2(r^2-a^2)^{\frac{1}{2}} - a^2b}{ac}\right],$$

$$\text{M}'\left[x_3 = \frac{-b(r^2-a^2)^{\frac{1}{2}} + a^2}{c}; \quad y_3 = \frac{-b^2(r^2-a^2)^{\frac{1}{2}} + a^2b}{ac}\right],$$

$$\text{M}\left[x_4 = \frac{-b(r^2-a^2)^{\frac{1}{2}} - a^2}{c}; \quad y_4 = \frac{-b^2(r^2-a^2)^{\frac{1}{2}} - a^2b}{ac}\right];$$

hence $\quad x_1-x_2=-\dfrac{2a^2}{c}; \quad x_3-x_4=\dfrac{2a^2}{c}; \quad y_1-y_2=y_3-y_4=\dfrac{2ab}{c};$

therefore $\quad \mathrm{LL}'=\left\{(x_1-x_2)^2+(y_1-y_2)^2\right\}^{\frac{1}{2}}=\left\{(x_3-x_4)^2+(y_3-y_4)^2\right\}^{\frac{1}{2}}$
$$=\mathrm{MM}'=2a=\text{major axis}.$$

The equations of tangents to the circle at L, L′ are respectively

$$(r^2-a^2)^{\frac{1}{2}}(bx+ay+bc)=a(ax-by+ac)\ \dots\dots\dots\dots\dots\dots (5),$$

$$(r^2-a^2)^{\frac{1}{2}}(bx+ay-bc)=a(by-ax+ac)\ \dots\dots\dots\dots\dots\dots (6).$$

Therefore the coordinates of the foci of the hyperbola will satisfy (5), (6); and eliminating $(r^2-a^2)^{\frac{1}{2}}$ from (5), (6), we have, for the locus of points of intersection of these tangents,

$$abx^2+(a^2-b^2)xy-aby^2-abc^2=0\ \dots\dots\dots\dots\dots\dots (7),$$

whose asymptotes are $\quad y=\dfrac{a}{b}x, \quad -y=-\dfrac{b}{a}x;$

so that (7) will be the equation of a rectangular hyperbola, one of whose asymptotes coincides with one asymptote of the fixed hyperbola (1). And since (7) is satisfied by the coordinates of the foci ($\pm c$, 0), this hyperbola passes through the foci of (1).

The abscissa of the middle point of LM is $-2a^2c^{-1}$; hence one directrix bisects LM, the other bisects L′M′.

The equation of LM is $\quad b^2(r^2-a^2)+acy(r^2-a^2)^{\frac{1}{2}}-a^2(a^2+cx)=0\ \dots (8)$, and as this equation contains the indeterminate $(r^2-a^2)^{\frac{1}{2}}$ in the 2nd degree, LM will envelop a conic, whose equation we obtain from the condition that (8) shall have equal roots; thus the envelop will be the parabola

$$y^2=\frac{4b^2}{c^2}(a^2+cx), \quad \text{or, with another origin,} \quad y^2=\frac{4b^2}{c}x.$$

The abscissa of the vertex, with reference to the centre of the hyperbola, is a^2c^{-1}. Therefore the tangent at the vertex of the parabola is one directrix of the hyberbola (1), and the focus of the parabola is at a distance c from the centre of the hyperbola (1); hence it coincides with one of the foci of the hyperbola (1).

5192. (By H. T. Gerrans, B.A.)—AB is a fixed diameter of a circle, OA a chord, ON an ordinate of the diameter, AP a line drawn so that $\angle$OAP $=\angle$OAN, and AP$=$AN; find the locus of P.

Solution by R. E. Riley, B.A.; E. Rutter, *and many others.*

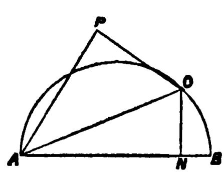
Let $\angle$OAP $=\angle$OAN $=\theta$, and $a=$ the radius of the circle; then we have

$$\mathrm{AP}=\mathrm{AN}=a(1+\cos 2\theta);$$

therefore the locus is of the form

$$r=a(1+\cos\phi),$$

and is, therefore, the cardioid.



5111. (By Professor Wolstenholme, M.A.)—1. If α, β be two angles such that

$$[1+2(\cos\alpha)^{\frac{2}{3}}][1+2(\cos\beta)^{\frac{2}{3}}] = 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{A}),$$

prove that

$$\frac{(1-8\cos^5\alpha)^{\frac{3}{2}}}{\sin^3\alpha\cos\alpha} = \frac{(1+8\cos^5\beta)^{\frac{3}{2}}}{\sin^3\beta\cos\beta}.$$

2. A circle and a rectangular hyperbola each passes through the centre of the other, and α, β are the two acute angles of intersection of the curves at their two real common points; prove that α, β will satisfy the equation (A), and that the squares of their latera recta are in the ratio

$$(1+8\cos^5\alpha)^{\frac{3}{2}} : 8\sin^3\alpha\cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{B}).$$

3. If a circle and a parabola be such that the circle passes through the focus of the parabola, and its centre lies on the directrix, prove that their angles of intersection satisfy the equation (A), and their latera recta are in the ratio (B).

4. If a rectangular hyperbola and a parabola be such that the centre of the hyperbola is the focus of the parabola, and the directrix of the parabola touches the hyperbola; then, if their acute angles of intersection be $\pi-2\alpha$, $\pi-2\beta$, prove that α, β will satisfy the equation (A), and that the squares on the latera recta are in the ratio (B).

[Professor Wolstenholme remarks that the three curves form an harmonic system such that the polar reciprocal of any one with respect to a second is the third, and that for any two the relations $\Theta=0$, $\Theta'=0$ hold; so that triangles can be drawn which are inscribed to one, circumscribed to a second, and self-conjugate to the third, in any order. The equations of three such curves may always, in two ways, be reduced to the forms $x^5+2yz=0$, $y^5+2zx=0$, $z^5+2xy=0$.]

Solution by the Proposer.

1. Let $1+2(\cos\alpha)^{\frac{2}{3}} = k$, $1+2(\cos\beta)^{\frac{2}{3}} = k'$; then $kk'=3$, and

$$\frac{(1+8\cos^2\alpha)^3}{8\cos^2\alpha\sin^6\alpha} = \frac{[1+(k-1)^3]^3}{(k-1)^3[1-\frac{1}{8}(k-1)^3]^3} = \frac{512(k^3-3k^2+3k)^3}{(k-1)^3(9-3k+3k^2-k^3)^3}$$

$$= \frac{512k^6(k-3+3k^{-1})^3}{(k-1)^3(3-k)^3(3+k^2)^3} = \frac{512(k-3+3k^{-1})^3}{(4-k-3k^{-1})^3(k+3k^{-1})^3},$$

or is a function of $k+3k^{-1}$ only, and is unaltered by substituting $3k^{-1}$ for k, or $\quad = (1+8\cos^2\beta)^3 \div 8\cos^2\beta\sin^6\beta$.

2. If we have a circle and rectangular hyperbola, each of which passes through the centre of the other, draw through the centre of the circle two straight lines parallel to the asymptotes of the hyperbola. These straight lines and the line at infinity will form a triangle whose angular points lie on the hyperbola, and which is self-conjugate to the circle. Similarly, if we join the centre of the hyperbola to the two points at infinity on the circle, we get a triangle inscribed in the circle and self-conjugate to the hyperbola. Hence the invariants Θ, Θ' of the two curves vanish, and therefore also the equations $\Theta^2 = 4\Theta'\Delta$, $\Theta'^2 = 4\Theta\Delta'$ are satisfied, so that we can inscribe in one triangles which either circumscribe the other or are self-conjugate to the other, and we can also circumscribe to one triangles which are either inscribed in the other or self-conjugate to it. If the equations of the two referred to the common self-conjugate triangle

be $lx^2+my^2+nz^2=0$, $l'x^2+m'y^2+n'z^2=0$, we shall have

$$\frac{l}{l'}+\frac{m}{m'}+\frac{n}{n'}=0,\quad \frac{l'}{l}+\frac{m'}{m}+\frac{n'}{n}=0,\quad \text{or}\quad \frac{l'}{l}=\frac{m'}{\omega m}=\frac{n'}{\omega^2 n},$$

where $\omega^2+\omega+1=0$, or ω is an impossible cube root of unity. The reciprocal of either with respect to the other will then be

$$lx^2+\omega^2 my^2+\omega nz^2=0,$$

so that we shall have the system

$$lx^2+my^2+nz^2=0,\quad lx^2+\omega my^2+\omega^2 nz^2=0,\quad lx^2+\omega^2 my^2+\omega nz^2=0,$$

for any pair of which the relations $\Theta=0$, $\Theta'=0$ hold, with the consequences already deduced.

The harmonic locus and harmonic envelope for any two coincide with the third, so that any tangent to one is divided harmonically by the other two, and the tangents drawn from any point of one to the other two form a harmonic pencil.

If we draw a parabola with its focus at the centre of the hyperbola and directrix touching the hyperbola at the centre of the circle, we see that it touches the sides of both the triangles before mentioned; and if we take the triangle formed by the asymptotes and the tangent at the centre of the circle, we see that this triangle will be inscribed to the circle, circumscribed to the hyperbola, and self-conjugate to the parabola. Hence for the parabola and either of the other curves we shall have $\Theta=0$, $\Theta'=0$, and the parabola will be the third curve of the system. Next, let C be a common point of the circle and rectangular hyperbola, and CA, CB tangents to the circle and hyperbola meeting the hyperbola and circle in A, B. Then, by considering the vanishing triangles circumscribed to the circle (or hyperbola), inscribed to the hyperbola (or circle), and self-conjugate to the parabola, we see that AB will touch the hyperbola and circle at A, B, and that the parabola will touch CA, CB at A, B. Similarly with the other real common point C′ we shall get another triangle A′B′C′ such that the circle touches A′B′, A′C′ at B′, C′, the hyperbola touches B′C′, B′A′ at C′, A′, and the parabola touches C′A′, C′B′ at A′, B′. The equations of the three curves referred to the triangle ABC will then be of the form

$$x^2+2pyz=0,\quad y^2+2qzx=0,\quad z^2+2rxy=0;$$

but the equation of the reciprocal of the first with respect to the second, or of the second with respect to the first, is

$$z^2+\frac{2}{pq}xy=0,$$

whence we must have $pqr=1$. Thus the equations of the system may be written $x^2+2yz=0$, $y^2+2zx=0$, $z^2+2xy=0$, and that in two ways according as we take the triangle ABC or A′B′C′. (Properly speaking, there are four ways, but only two are real.)

If, however, we take areal coordinates on the triangle ABC, the three equations $\quad x^2+2pyz=0,\quad y^2+2qzx=0,\quad z^2+2rxy=0$ .... (1, 2, 3) will represent a circle, rectangular hyperbola, and parabola, if

$$b^2=c^2,\quad p=\frac{-a^2}{2b^2}=-2\cos^2 B\equiv -2\cos^2\alpha \text{ ....(1)};$$

$$q=\frac{b^2}{2ab\cos B}=\frac{1}{4\cos^2\alpha};\quad r=-2 \text{ ....(2, 3)};$$

so that the condition $pqr=1$ is satisfied.

Where (1) again meets (2), $qx^3=py^3$, or $8\cos^4\alpha\, x^3=-y^3$; or, if

$\cos\alpha = k^3$, $x = -2k^4y$, and the common point is $\dfrac{x}{-2k^4} = y = \dfrac{z}{k^2}$. The tangents to (1), (2) at this point are

$$x + k^4y + k^2z = 0, \quad \text{and} \quad x + 4k^4y - 2k^2z = 0 \; \ldots\ldots\ldots \; \text{(I., II.)};$$

whence $\cos\beta$ = a fraction whose numerator is

$$\sin^2 A + 4k^8 \sin^2 B - 2k^4 \sin^2 C - 2k^6 \sin B \sin C \cos A$$
$$+ k^2 \sin C \sin A \cos B - 5k^4 \sin A \sin B \cos C,$$

and denominator the product of the two radicals

$$\{\sin^2 A + k^8 \sin^2 B + k^4 \sin^2 C - 2k^6 \sin B \sin C \cos A$$
$$- 2k^2 \sin C \sin A \cos B - 2k^4 \sin A \sin B \cos C\}^{\frac{1}{2}},$$
$$\{\sin^2 A + 16k^8 \sin^2 B + 4k^4 \sin^2 C + 16k^6 \sin B \sin C \cos A$$
$$+ 4k^2 \sin C \sin A \cos B - 8k^4 \sin A \sin B \cos C\}^{\frac{1}{2}}.$$

But $\sin A = \sin 2\alpha = 2k^3 \sin\alpha, \quad \cos A = 1 - 2k^6, \quad B = C = \alpha,$

$$\sin B \sin C \cos A = (1 - 2k^6) \sin^2\alpha,$$
$$\sin C \sin A \cos B = \sin A \sin B \cos C = 2k^6 \sin^2\alpha;$$

and, on making these substitutions, the numerator becomes

$$2k^4 (k^2 - 1)^3 (2k^2 + 1) \sin^2\alpha,$$

and the two radicals

$$\{k^4 (1 - k^2)^2 (2k^2 + 1)^2 \sin^2\alpha\}^{\frac{1}{2}}, \quad \{4k^4 (1 + 2k^2)^3 (1 - k^2) \sin^2\alpha\}^{\frac{1}{2}};$$

whence $\cos\beta = \left(\dfrac{1 - k^2}{2k^2 + 1}\right)^{\frac{3}{2}}$, and $1 + 2(\cos\beta)^{\frac{2}{3}} = 1 + \dfrac{2(1 - k^2)}{1 + 2k^2} = \dfrac{3}{1 + 2k^2}$,

or $$\{1 + 2(\cos\beta)^{\frac{2}{3}}\}\{1 + 2(\cos\alpha)^{\frac{2}{3}}\} = 3.$$

Now the ratio of the areas of the two curves $Px^2 + 2pyz = 0$, $Qy^2 + 2qzx = 0$ is $\dfrac{Pp^2}{(p^2 + 2Pp)^{\frac{3}{2}}} : \dfrac{Qq^2}{(q^2 + 2Qq)^{\frac{3}{2}}}$; hence the squares of the latera-recta of the circle and hyperbola will be as

$$\frac{p^{\frac{1}{2}}}{(2 + p)^{\frac{3}{2}}} : \frac{q^{\frac{1}{2}}}{(-q - 2)^{\frac{3}{2}}}, \quad \text{or} \quad \frac{(-2\cos^2\alpha)^{\frac{1}{2}}}{(2\sin^2\alpha)^{\frac{3}{2}}} : \frac{\left(\dfrac{1}{4\cos^2\alpha}\right)^{\frac{1}{2}}}{\left(-\dfrac{1}{4\cos^2\alpha} - 2\right)^{\frac{3}{2}}},$$

or $$(1 + 8\cos^2\alpha)^{\frac{3}{2}} : 8\sin^3\alpha\cos\alpha.$$

Of course, by using the triangle A′B′C′, we should get the equivalent ratio $(1 + 8\cos^2\beta)^{\frac{3}{2}} : 8\sin^3\beta\cos\beta$, which is a sufficient test.

3, 4. If l_1, l_2, l_3 be the latera-recta of the three curves, we have $l_1 l_3 = l_2^2$, since the semi-latus-rectum of the parabola is the perpendicular from the centre of the hyperbola on the tangent at a point whose central distance is $\frac{1}{2}l_1$. Hence $\dfrac{l_1}{l_2} = \dfrac{l_2}{l_3} = \left(\dfrac{l_1}{l_3}\right)^{\frac{1}{2}}$;

also the angle of intersection of the circle and parabola at B is α, and at B′ is β, whence $\quad l_1 : l_3 = l_1^2 : l_2^2$ = the ratio (B).

The angle of intersection of the parabola and hyperbola at A is $\pi - 2\alpha$, and at A′ is $\pi - 2\beta$, whence

$$l_2^2 : l_3^2 = l_1^2 : l_2^2 = \text{the ratio (B)}.$$

5224. (By the Rev. H. G. Day, M.A.)—On each of n pillars, whose heights, in ascending order of magnitude, are $c_1, c_2, c_3, \ldots, c_n$, points are taken at random; find the chance of the point so taken on the rth pillar being the highest.

Solution by the Proposer.

The chance that the highest point so taken is below the height c_s is $\dfrac{(c_s)^{n-s}}{c_{s+1}c_{s+2}\ldots c_n}$; and similarly for c_{s-1}.

Hence the chance that the highest point taken lies between the heights c_s and c_{s-1} is
$$\frac{(c_s)^{n-s}}{c_{s+1}c_{s+2}\ldots c_n}-\frac{(c_{s-1})^{n-s+1}}{c_s\,.\,c_{s+1}\,.\,\ldots c_n}.$$

Let this expression be denoted by p_s.

Now the chances in *this case* of any particular one of the points on the pillars marked from s to n being highest will be equal, each being $\dfrac{p_s}{n-s+1}$.

Therefore the probability required is
$$\frac{p_1}{n}+\frac{p_2}{n-1}+\frac{p_3}{n-2}+\ldots+\frac{p_r}{n-r+1}.$$

5504. (By Dr. Booth, F.R.S.)—If Δ be the area of a plane triangle, prove that $4\Delta^2 = abc\sigma$, where a, b, c are the sides of the original triangle, and σ the semiperimeter of its orthocentric triangle.

Solution by Christine Ladd, D. Edwardes, J. O'Regan, *and many others.*

It is easy to show that radii of the circumscribed circle drawn to the points A, B, C are perpendicular to the sides (a', b', c' suppose) of the orthocentric triangle; hence the triangle ABC is divided into three parts whose areas are respectively $\frac{1}{2}Ra'$, $\frac{1}{2}Rb'$, $\frac{1}{2}Rc'$; hence we have
$$\Delta = \tfrac{1}{2}R(a'+b'+c') = R\sigma = \frac{abc\sigma}{4\Delta},\quad \text{therefore}\quad 4\Delta^2 = abc\sigma.$$

132. Contraposition: by Alexander J. Ellis, F.R.S.

In the *Syllabus* of the Association for the Improvement of Geometrical Teaching (p. 4) we read:—

"In the typical Theorem, If A is B, then C is D ....(i.);
the hypothesis is that A is B, and the conclusion that C is D. From the truth conveyed in this Theorem it necessarily follows:

If C is not D, then A is not B ....(ii.)

Two such Theorems as (i.) and (ii.) are said to be *contrapositive*, each of the other."

At the meeting of the Association on January 11th, I was much surprised to hear the President read letters complaining of the difficulty of this piece of logic, one of which, so far as I can remember, asserted that it was impossible to prove it in the case of a negative proposition, that is, I suppose, to derive (i.) from (ii.). It seems, therefore, advisable to put the proof into an easy form, following the principles of Boole's Laws of Thought, but not his elaborate notation and theory.

We deal with four propositions, A is B, A is not B, C is D, C is not D. Now, as *either* the positive *or* the negative form of each must be true, these can co-exist in four different ways only, viz. :—

(1) A is B, and at the same time C is D ;
or (2) A is B, and at the same time C is not D ;
or (3) A is not B, and at the same time C is D ;
or, finally, (4) A is not B, and at the same time C is not D.

This merely states a fundamental relation of thought without any hypothesis.

Now suppose that by any course of reasoning it is established that "if (or whenever) A is B, then C is D." This makes (1) exist, but denies the existence of (2). Let the reader conceal the line (2) with a pencil, as *excluded* by the demonstration of a particular theorem. Then let him examine the lines (1), (3), and (4) for the cases in which C is not D. He will find this in (4) only, where it is associated with "A is not B." Hence it follows from the fact "whenever A is B, C is D," that "whenever C is *not* D, A is *not* B," or (ii.) is derived from (i.). Observe that it does *not* follow conversely that "whenever A is not B, then C is not D," because both of the lines (3) and (4) are left untouched by the original proposition.

Now take the negative proposition, "if C is not D, then A is not B." This excludes the possibility of "C is not D" co-existing with "A is B," that is, it excludes the same case (2) as before. Now, covering (2) with a pencil, look for "A is B" in lines (1), (3), and (4), and we only find it in (1) associated with "C is D." That is, under these circumstances, if A is B, then C is D; that is, (i.) follows from (ii.), just as (ii.) followed from (i.). This relation constitutes the fact known as contraposition. Observe however, again, that the converse, "if C is D, then A is B," does not follow from (ii.), for lines (1) and (3), which are left untouched, both contain "C is D," shewing that, so far as (ii.) is concerned, "C is D" may co-exist either with "A is B" or with "A is not B."

As many pupils have great difficulty with conversion as well as contraposition, this simple means of putting the four cases before them, and leading them to draw the inference themselves, may possibly be useful to them not only in geometry, but in all after life.

5466. (By the Editor.)—If two random points be taken one in each of (1) the arcs, (2) the areas, of two semicircles that together make up a complete circle; find, in each case, (α) the average distance between the points, and (β) the probability that this distance is less than the radius.

Solution by E. B. Seitz.

1. (α). Let ADB and ACB (Fig. 1) be the semicircles, M and N the random points, and O the centre of the circle.

Put $OA = r$, $\angle AOM = \theta$, and $\angle AON = \phi$; then we have $MN = 2r \sin \tfrac{1}{2}(\theta + \phi)$; also an element of the arc at M is $r\,d\theta$, and at N, $r\,d\phi$; moreover, the limits of θ are 0 and π, and of ϕ, 0 and π; hence the required average

$$= \frac{1}{\pi^2 r^2} \int_0^{\pi} \int_0^{\pi} 2r \sin \tfrac{1}{2}(\theta + \phi)\, r\,d\theta\, r\,d\phi$$

$$= \frac{4r}{\pi^2} \int_0^{\pi} (\sin \tfrac{1}{2}\theta + \cos \tfrac{1}{2}\theta)\, d\theta = \frac{16r}{\pi^2}.$$

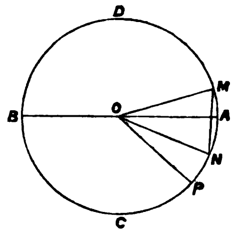


Fig. 1.

1. (β). Let the angle $MOP = \tfrac{1}{3}\pi$ (Fig. 1); then, while M is fixed, N may be taken anywhere in the arc AP, and the distance MN will be less than the radius; also the limits of θ are 0 and $\tfrac{1}{3}\pi$, and doubled: hence the

required probability $$= \frac{2}{\pi^2 r^2} \int_0^{\frac{1}{3}\pi} (\tfrac{1}{3}\pi - \theta)\, r^2\, d\theta = \frac{1}{9}.$$

2. (α). Draw the chord EF through M, N (Fig. 2), and draw OC perpendicular to EF. Put $OD = x$, $DM = y$, $DN = z$, $DE = y_1$, $DF = z_1$, $\angle ADM = \theta$, and $\angle OEC = \phi$; then we have

$$OC = x \sin \theta = r \sin \phi, \quad \sin \theta\, dx = r \cos \phi\, d\phi,$$

$$y_1 = (r^2 - x^2 \sin^2 \theta)^{\frac{1}{2}} - x \cos \theta = r(\cos \phi - \cot \theta \sin \phi),$$

$$z_1 = (r^2 - x^2 \sin^2 \theta)^{\frac{1}{2}} + x \cos \theta = r(\cos \phi + \cot \theta \sin \phi);$$

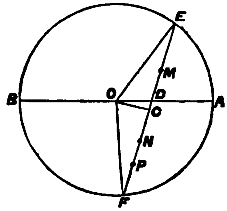


Fig. 2.

also an element at M is $\sin \theta\, dx\, dy$, or $r \cos \phi\, d\phi\, dy$, and at N, $(y + z)\, d\theta\, dz$; moreover, the limits of θ are 0 and π; of x, $-r$ and r; of y, 0 and y_1; and of z, 0 and z_1; hence, since the whole number of ways the two points can be taken is $\tfrac{1}{4}\pi^2 r^4$, the required average

$$= \frac{4}{\pi^2 r^4} \int_0^{\pi} \int_{-r}^{r} \int_0^{y_1} \int_0^{z_1} (y + z)^2 \sin \theta\, d\theta\, dx\, dy\, dz$$

$$= \frac{1}{3\pi^2 r^4} \int_0^{\pi} \int_{-r}^{r} [(y_1 + z_1)^4 - (y_1^4 + z_1^4)] \sin \theta\, d\theta\, dx$$

$$= \frac{2}{3\pi^2 r^4} \int_0^{\pi} \int_{-r}^{r} [7(r^2 - x^2)^2 + 8(r^2 - x^2)\, x^2 \cos^2 \theta] \sin \theta\, d\theta\, dx$$

$$= \frac{32r}{45\pi^2} \int_0^{\pi} (7 + 2\cos^2 \theta) \sin \theta\, d\theta = \frac{1472r}{135\pi^2}.$$

2. (β). Let MP = the radius (Fig. 2); then, while M is fixed, N may be taken anywhere in DP, and the distance MN will be less than the

radius. The limits of θ are 0 and $\frac{1}{2}\pi$, and doubled; and the limits of ϕ are 0 and θ, and doubled. When $\theta < \frac{1}{3}\pi$, the limits of y are 0 and y_1, and of z, 0 and $r-y$. From $\theta = \frac{1}{3}\pi$ to $\theta = \frac{1}{2}\pi$ the limits of y are 0 and y_1 and of z, 0 and $r-y$, when $\phi < \pi - 2\theta$; when $\phi > \pi - 2\theta$ and $< \frac{1}{3}\pi$, the limits of z are 0 and z_1 from $y=0$ to $y = r - z_1$, and they are 0 and $r-y$ from $y = r - z_1$ to $y = y_1$; and when $\phi > \frac{1}{3}\pi$, the limits of y are 0 and y_1, and of z, 0 and z_1. Hence the required probability

$$= \frac{16}{\pi^2 r^4}\int_0^{\frac{1}{3}\pi}\int_0^{\theta}\int_0^{y_1}\int_0^{r-y} r(y+z)\,d\theta\cos\phi\,d\phi\,dy\,dz$$

$$+ \frac{16}{\pi^2 r^4}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left\{\int_0^{\pi-2\theta}\int_0^{y_1}\int_0^{r-y} r(y+z)\cos\phi\,d\phi\,dy\,dz\right.$$

$$+ \int_{\pi-2\theta}^{\frac{1}{3}\pi}\left[\int_0^{r-z_1}\int_0^{z_1} r(y+z)\,dy\,dz + \int_{r-z_1}^{y_1}\int_0^{r-y} r(y+z)\,dy\,dz\right]\cos\phi\,d\phi$$

$$\left. + \int_{\frac{1}{3}\pi}^{\theta}\int_0^{y_1}\int_0^{z_1} r(y+z)\cos\phi\,d\phi\,dy\,dz\right\}d\theta$$

$$= \frac{8}{3\pi^2}\int_0^{\frac{1}{3}\pi}\int_0^{\theta}\left[3\operatorname{cosec}\theta\sin(\theta-\phi) - \operatorname{cosec}^3\theta\sin^3(\theta-\phi)\right]d\theta\cos\phi\,d\phi$$

$$+ \frac{8}{3\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left\{\int_0^{\pi-2\theta}\left[3\operatorname{cosec}\theta\sin(\theta-\phi) - \operatorname{cosec}^3\theta\sin^3(\theta-\phi)\right]\cos\phi\,d\phi\right.$$

$$+ \int_{\pi-2\theta}^{\frac{1}{3}\pi}\left[6\cos\phi - 6\cot^2\theta\sin^2\phi\cos\phi - 2\cos^3\phi - 2\right]\cos\phi\,d\phi$$

$$\left. + \int_{\frac{1}{3}\pi}^{\theta}(6\cos^2\phi - 6\cot^2\theta\sin^2\phi)\cos^2\phi\,d\phi\right\}d\theta$$

$$= \frac{1}{\pi^2}\int_0^{\frac{1}{3}\pi}(4\theta - \theta\operatorname{cosec}^2\theta + \cot\theta)\,d\theta$$

$$+ \frac{1}{\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(16\theta - 4\pi + \pi\operatorname{cosec}^2\theta - 4\theta\operatorname{cosec}^2\theta + 4\cot\theta - 3\sqrt{3} + 8\sin\theta\cos\theta)\,d\theta$$

$$= \frac{2}{3} - \frac{\sqrt{3}}{2\pi}.$$

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133. TO FIND THE DIRECTRIX OF THE PARABOLA $(ax+by)^2 + 2dx + 2ey + f = 0$. *By* W. GALLATLY, B.A.

If T be any point on the directrix, then the tangents through T are at right angles. Now, let a radius-vector be drawn through T to the curve, cutting it in P, Q; and let θ be the inclination of TP to the axis of x, TP be r, P be (xy), and T be (hk); then

$$x = h + r\cos\theta, \quad \text{and} \quad y = k + r\sin\theta;$$

$$\therefore\ \left\{r(a\cos\theta+b\sin\theta)+ah+bk\right\}^2+2r(d\cos\theta+e\sin\theta)+2dh+2ek+f=0;$$

$$\therefore\ r^2(a\cos\theta+b\sin\theta)^2+2r\left\{(ah+bk)(a\cos\theta+b\sin\theta)+(d\cos\theta+e\sin\theta)\right\}$$
$$+2dh+2ek+f+(ah+bk)^2=0.$$

If TPQ be a tangent, we have

$$(a\cos\theta+b\sin\theta)^2\left\{(ah+bk)^2+2dh+2ek+f\right\}$$
$$=\left\{(ah+bk)(a\cos\theta+b\sin\theta)+d\cos\theta+e\sin\theta\right\}^2;$$

$$\therefore\ (a\cos\theta+b\sin\theta)^2(2dh+2ek+f)$$
$$=2(ah+bk)(a\cos\theta+b\sin\theta)(d\cos\theta+e\sin\theta)+(d\cos\theta+e\sin\theta)^2;$$

$$\therefore\ (a+b\tan\theta)^2(2dh+2ek+f)$$
$$=2(ah+bk)(a+b\tan\theta)(d+e\tan\theta)+(d+e\tan\theta)^2.$$

But if θ_1, θ_2 are the two angles thus found,

$$1+\tan\theta_1\tan\theta_2=0;$$

$$\therefore\ b^2(2dh+2ek+f)-2be(ah+bk)-e^2+a^2(2dh+2ek+f)$$
$$-2ad(ah+bk)-d^2=0;$$

therefore the directrix is

$$2bx(bd-ae)+2ay(ae-bd)+b^2f-e^2-d^2+a^2f=0,$$

or
$$bx-ay=\frac{e^2+d^2-(a^2+b^2)f}{2(bd-ae)}.$$

5320. (By J. J. Walker, M.A.) — If normals to the ellipse $b^2x^2+a^2y^2-a^2b^2=0$ be drawn from any point on the curve

$$(a^2x^2+b^2y^2-c^4)^3+54a^2b^2c^4x^2y^2=0,$$

prove that they form an harmonic pencil.

Solution by C. Leudesdorf, M.A.

The four points where the normals drawn from (X, Y) to the ellipse cut the ellipse are given as its intersections with the conic

$$c^2xy-a^2y\mathrm{X}+b^2x\mathrm{Y}=0.$$

Eliminating y between this and the equation to the ellipse, we find

$$c^4x^4-2c^2a^2x^3\mathrm{X}+(b^2\mathrm{Y}^2+a^2\mathrm{X}^2-c^4)a^2x^2+2c^2a^4\mathrm{X}x-a^6\mathrm{X}^2=0\ \ldots\ldots(1).$$

Since the equation to the normal at (x', y') is $a^2xy'-b^2yx'=c^2x'y'$, the roots of (1) are $\frac{a^2}{c^2}$ times the intercepts made on the axis of x by the four normals that can be drawn through (X, Y). If, then, these normals form a harmonic pencil, the invariant T of (1) must vanish, so that

$$-2a^6(b^2\mathrm{Y}^2+a^2\mathrm{X}^2-c^4)^3+108a^{10}c^4\mathrm{X}^4-108a^8c^8\mathrm{X}^2$$
$$-36a^8c^4\mathrm{X}^2(b^2\mathrm{Y}^2+a^2\mathrm{X}^2-c^4)-72a^8c^4\mathrm{X}^2(b^2\mathrm{Y}^2+a^2\mathrm{X}^2-c^4)=0,$$

which reduces to $(a^2\mathrm{X}^2+b^2\mathrm{Y}^2-c^4)^3+54a^2b^2c^2\mathrm{X}^2\mathrm{Y}^2=0.$

5173. (By H. T. GERRANS, B.A.)—Find the sums of the infinite series
$$\frac{x^0}{2}+\frac{x^6}{5}+\frac{x^{12}}{8}+\frac{x^{18}}{11}+\frac{x^{24}}{14}+\&c., \quad \frac{x^3}{3}-\frac{x^5}{3.5}+\frac{x^7}{5.7}-\frac{x^9}{7.9}+\&c.$$

Solution by Rev. J. L. KITCHIN, M.A.; R. TUCKER, M.A.; *and others.*

1. If $S=\frac{x^2}{2}+\frac{x^5}{5}+\frac{x^8}{8}+\ldots$, $\frac{dS}{dx}=x+x^4+x^7+\ldots=\frac{x}{1-x^3}$;

$$\therefore S=\int\frac{x\,dx}{1-x^3}=\tfrac{1}{3}\left[\tfrac{1}{2}\log(x^2+x+1)-\log(1-x)-\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]+C;$$

and when $x=0$, $S=0$; therefore $C=\frac{1}{18}\pi\sqrt{3}$.

Now the given series $=\frac{1}{x^4}\left[\frac{x^4}{2}+\frac{x^{10}}{5}+\frac{x^{16}}{8}+\ldots\right]$

$$=\frac{1}{x^4}\left[\tfrac{1}{3}\left\{\tfrac{1}{2}\log(x^4+x^2+1)-\log(1-x^2)-\sqrt{3}\tan^{-1}\frac{2x^2+1}{\sqrt{3}}+\frac{\pi\sqrt{3}}{6}\right\}\right].$$

2. $2\Sigma=x^3(1-\frac{1}{3})-x^5(\frac{1}{3}-\frac{1}{5})+\ldots=(x^2+1)\tan^{-1}x-x.$

5101. (By A. MARTIN, M.A.)—An auger-hole is made through the centre of a sphere; show that the average of the volume removed is, in parts of the volume of the sphere, $1-\frac{3}{16}\pi$.

Solution by the PROPOSER.

Let x = radius of auger-hole, r = radius of the sphere, and V = volume removed. The volume removed consists of a cylinder, radius x and altitude $2(r^2-x^2)^{\frac{1}{2}}$, and two equal spherical segments, base diameters $2x$ and heights $r-(r^2-x^2)^{\frac{1}{2}}$. The volume of the cylinder is $2\pi x^2(r^2-x^2)^{\frac{1}{2}}$, and the volume of the two segments is

$$\pi x^2\left\{r-(r^2-x^2)^{\frac{1}{2}}\right\}+\tfrac{1}{3}\pi\left\{r-(r^2-x^2)^{\frac{1}{2}}\right\}^3;\ \text{therefore } V=\tfrac{4}{3}\pi\left\{r^3-(r^2-x^2)^{\frac{3}{2}}\right\}.$$

The *average* volume removed $=\frac{\int V dx}{\int dx}=\frac{1}{r}\int_0^r V dx=\frac{4}{3}\pi r^3(1-\frac{3}{16}\pi).$

5367. (By ELIZABETH BLACKWOOD.)—If X, Y, Z be random points taken respectively in a sphere, in a great circle of the sphere, in a radius of the sphere; show that the respective chances of X, Y, Z being farthest from the centre are as 3, 2, 1.

Solution by the PROPOSER.

The problem may be enunciated analytically as follows:—Given that

all values of x^3, y^2, z between 0 and 1 are equally probable, and no other values possible, find the respective chances of x, y, or z being the greatest of the three variables. The respective chances are easily seen to be

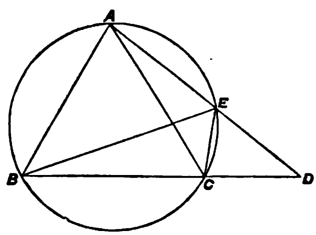
$$\int_0^1 d(x^3)\int_0^x d(y^2)\int_0^x dz = \tfrac{1}{2},\quad \int_0^1 d(y^2)\int_0^y d(x^3)\int_0^y dz = \tfrac{1}{3},$$

$$\int_0^1 dz\int_0^z d(x^3)\int_0^z d(y^2) = \tfrac{1}{6}.$$

5445. (By C. H. Pillai)—A point D is taken in the side BC produced of an equilateral triangle ABC, and CE is drawn cutting AD in E, and making the angle ACE = ADC : prove that

(1) AE + EC = BE; (2) AE . EC = $\overline{BE}^2 \sim \overline{BC}^2$; (3) AE : EC = BC : CD.

Solution by J. O'Regan; A. W. Cave; *and many others.*



1. Since we have

$\angle$BCA = CDA + CAD
= ACE + CAE = 60°,

therefore $\angle$ AEC = 120°, and a circle will pass through A, B, C, E; hence [by a well-known property, which the solvers prove readily from Euc. VI. D], we have BE = AE + CE.

2. $\overline{BE}^2 - \overline{BC}^2 = (BE + EC)^2 - \overline{AC}^2$
$= (\overline{AE}^2 + \overline{EC}^2 + 2AE\,.\,EC)$
$-(\overline{AE}^2 + \overline{EC}^2 + AE\,.\,EC) = AE\,.\,EC.$

3. Since $\angle$ ACE = ADC, and the angle CAE is common to the two triangles ACE and ACD, the two triangles are similar, and therefore

AE : EC = AC : CD = BC : CD.

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133. On the Random Chord Question: by Helen Thomson. [See *Reprint*, Vol. XXVIII., pp. 106—110, Vol. XXIX., pp. 17—20.]

Unto Fitz-James made answer Roderick Dhu,
"Your hasty threat you will as quickly rue.
To fence in logic is 't that you aspire?
Abracadabra! why, you 're in the mire.
A joust of this sort you can ill afford.
Here 's 'random' attribute of subject 'chord';
What was 't you took at random? Points, you said.
Then where 's your logic? 'Tis knocked on the head.
My line 's at random, and my line 's a chord."
Thus Roderick broke Fitz-James's vaunted sword.

5480. (By NILKANTHA SARKAR, B.A.)—C is the centre of an ellipse, CB its semi-minor axis; also a circle is drawn concentric with the ellipse, and touching its two directrices, and meeting CB produced in A; determine (1) the eccentricity of the ellipse in order that CA may be bisected in B, and (2) whether the ellipse $2x^2+3y^2=c^2$ answers the condition.

Solution by SIMONELLI RUGGERO; CECCHINO VINCENZO; *and others.*

Sia R il raggio del cerchio. Rappresentandoci questo raggio la distanza del centro C del ellisse, dalle direttrici, avremo $R = a\ (a^2-b^2)^{-\frac{1}{2}}$.

Supponendo preso ad arbitrio l'asse $2a$, avremo per le condizioni del problema, $b = \frac{1}{2}R$, quindi $b^2 = \frac{1}{2}a^2$, che sostituito nell' espressione dell' eccentricità, avremo $e = \frac{1}{2}\sqrt{2}$.

Perciò l'eccentricità di tutte le infinite elissi che soddisfano alle condizioni richieste, è costante ed egua le ad $\frac{1}{2}\sqrt{2}$.

L'equazione proposta non soddisfa al problema, essendo $e = \frac{1}{3}\sqrt{3}$.

5569. (By Professor TANNER, M.A.)—Solve the functional equation

$$\phi(xy) = x\phi(y) + y\phi(x).$$

I. *Solution by* Prof. MOREL; Prof. EVANS, M.A.; *and others.*

Cette égalité devant avoir lieu quelsque soient x et y, faisons successivement $y = x,\ x^2, \ldots\ldots x^{n-1}$; il vient

$$\phi(x^2) = x\phi(x) + x\phi(x) = 2x\phi(x),$$
$$\phi(x^3) = x\phi(x^2) + x^2\phi(x) = 2x^2\phi(x) + x^2\phi(x) = 3x^2\phi(x),$$
$$\phi(x^4) = x\phi(x^3) + x^3\phi(x) = 3x^3\phi(x) + x^3\phi(x) = 4x^3\phi(x),$$

et en général $\qquad \phi(x^n) = nx^{n-1}\phi(x).$

Multiplions cette dernière égalité par x, et nous aurons

$$x\phi(x^n) = nx^n\phi(x). \quad \text{On peut l'écrire} \quad \frac{\phi(x^n)}{x^n} = n\,.\,\frac{\phi(x)}{x}.$$

Posons maintenant $x = a^z$, il vient $\dfrac{\phi(a^{zn})}{a^{zn}} = n\dfrac{\phi(a^z)}{a^z}$.

Mais le premier membre, dans lequel n et z sont quelconques, ne change pas, quand on change n en z, et z en n. On en déduit

$$\frac{n\phi(a^z)}{a^z} = \frac{z\phi(a^n)}{a^n}.$$

Comme le multiplicateur de z est constant, et que d'autre part on a $z = \log_a x$, on en déduit $\phi(x) = cx\log_a x$.

Réciproquement, je dis que toute fonction de la forme $cx \log_a x$ répond à la question. Car on a $cxy \log xy = cy \cdot x \log x + cx \cdot y \log y$; ce qui donne bien la relation $\phi(xy) = x\phi(y) + y\phi(x)$.

[Professor TANNER's solution is $\phi(x) = Ax \log x + \Sigma_i A_i \cdot \frac{dx}{dt_i}$,* where A, A_i, t_i are quantities that are the same for all subjects of ϕ, and the summation includes as many values of i as you please.]

II. *Solution by* Prof. TOWNSEND, F.R.S.; R. E. RILEY, B.A.; *and others.*

Differentiating with respect to x and to y successively, and eliminating $\phi'(xy)$, we get, immediately,

$$\phi'(x) - \frac{1}{x}\phi(x) = \phi'(y) - \frac{1}{y}\phi(y),$$

which, being true for all values of x and y, shows that each side = an arbitrary constant a; hence, at once, by integration,

$$\phi(x) = ax \log x, \quad \phi(y) = ay \log y,$$

therefore $\phi(xy) = axy \log xy = axy(\log x + \log y) = y\phi(x) + x\phi(y)$, the constant of integration vanishing by virtue of the given relation.

III. *Solution by* J. L. MACKENZIE, J. O'REGAN, *and others.*

$$\frac{\phi(xy)}{xy} = \frac{\phi(x)}{x} + \frac{\phi(y)}{y}, \quad \text{or} \quad \psi(xy) = \psi(x) + \psi(y).$$

Let $x = e^\theta$, $y = e^\phi$; then we have

$$\psi(e^{\theta+\phi}) = \psi(e^\theta) + \psi(e^\phi), \quad \text{or} \quad f(\theta+\phi) = f(\theta) + f(\phi).$$

Differentiating with respect to θ, $f'(\theta+\phi) = f'(\theta)$;
Similarly $f'(\theta+\phi) = f'(\phi)$; hence $f'(\theta) = f'(\phi)$, or $f'(\theta) = k$, where k is a constant. Hence $f(\theta) = k\theta + k'$; and from the equation

$$f(\theta+\phi) = f(\theta) + f(\phi) \text{ we see that } k' = 0.$$

Hence $f(\theta) = k\theta$, $\psi(x) = k \log x$, and finally $\phi(x) = kx \log x$.

5456. (By Professor MINCHIN, M.A.)—If a body P moves in a plane orbit, so that the direction of its resultant acceleration is always a tangent to a given curve, prove that, if Q is the point of contact of this tangent, p the perpendicular from Q on the tangent at P, ω the angle subtended at P by the radius of curvature at Q, θ the angle which PQ makes with a fixed line, and h a constant, $vp = he^{\int \tan \omega \, d\theta}$.

[* On this Mr. MACKENZIE remarks that he believes that "no such addition as Prof. TANNER has here given to the value of $\phi(x)$ can be correct, except on the supposition that x is not an independent variable, but is limited to certain periodical values."]

I. *Solution by* J. C. MALET, M.A.; E. B. ELLIOTT, M.A.; *and others.*

Let P, P_1 be two positions of the moving particle, and Q, Q_1 the corresponding points on the curve enveloped by the direction of the acceleration; let R be the intersection of the lines PQ, P_1Q_1; then the difference of the moments of the velocities at P and P_1 about R is equal to

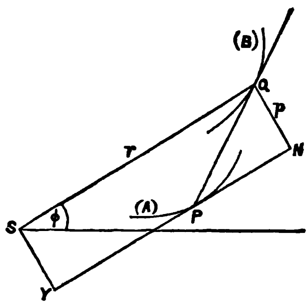
$$v_1p_1\frac{P_1R}{P_1Q_1}-vp\frac{PR}{PQ},\ \text{ or }\ v_1p_1\left(1-\frac{Q_1R}{P_1Q_1}\right)-vp\left(1+\frac{RQ}{PQ}\right).$$

This expression must vanish in the limit, hence we have

$$d(vp)-vp\frac{Q_1R+RQ}{PQ}=0,\ \text{ or }\ d(vp)-vp\frac{ds}{PQ}=0,$$

ds being the element of the curve QQ_1. Hence, integrating, we get required expression for vp.

II. *Solution by* R. F. DAVIS, B.A.; J. J. WALKER, M.A.; J. HAMMOND, M.A.; *and others.*



Denoting the curves by (A), (B), it is at once evident that (B) is hodographic to (A). Consequently, if QS be drawn parallel to the tangent at P, there is a fixed point S on this line, the radii vectores from which to (B) represent in magnitude and direction the velocities of the particle at the points in which corresponding tangents to (B) meet (A). Let θ, ϕ be the angles which the tangents at Q, P make with a fixed initial line through S; and SQ (which $\propto v$) $= r$. Then, if SY, QN be perpendiculars on the tangent at P, SY $=$ QN $= p$; hence $PY=\dfrac{dp}{d\phi}$.

Moreover $\qquad PN = p\cot QPN = p\cot SQP = \dfrac{p\,dr}{r\,d\phi}.$

Hence $\qquad \dfrac{dp}{d\phi}+\dfrac{p\,dr}{r\,d\phi}=r^2;\ \text{ or }\ \dfrac{d}{d\phi}(\log p)+\dfrac{d}{d\phi}(\log r)=\dfrac{r}{p}.$

But $\qquad p:PQ=r\,d\phi:ds=r\,d\phi:\rho\,d\theta,$

so that $\qquad \dfrac{r}{p}=\dfrac{\rho}{PQ}\cdot\dfrac{d\theta}{d\phi}=\tan\omega\cdot\dfrac{d\theta}{d\phi}.$

Therefore $\qquad pr=Ce^{\int\tan\omega\,d\theta};\ \text{ and } r\propto v$, &c.

5534. (By CHRISTINE LADD.)—If four conics S, A, B, C, have one focus and one tangent, D, common to all, and if a tangent common to S and A intersects D on a directrix of A, a tangent common to S and B on

a directrix of B, and a tangent common to S and C on a directrix of C, prove that the common tangents of A, B, C will meet in three points in a straight line.

Solution by D. Lo. Gatto; V. Jacobini; *and others.*

Se si hanno quattro coniche, s, a, b, c, aventi una tangente comune m, ed un fuoco comune F, e si chiamano d_a, d_b, d_c le direttrici delle 3 coniche a, b, c, corrispondenti al fuoco F, e t_a, t_b, t_c le tangenti comuni alle coniche s, a, s, b, s, c, se le terne di rette: $m\,t_a\,t_a$, $md_b t_b$, $md_c t_c$ concorrono in un punto, concorreranno pure in un punto le tre tangenti di a, b, c.

Prendiamo il noto teorema: Abbassando da un punto M sui lati a_2, b_2, c_2 di un triangolo le tre perpendicolari, se i piedi t_{a_2}, t_{b_2}, t_{c_2} di queste sono in linea retta, allora i vertici del triangolo si trovano sopra una circonferenza di cerchio passante per M. Trasformiamo questo teorema coll' inversione quadratica, scegliendo come centro di inversione il punto M. Allora il punto M, e le normali abbassate da M sui lati del triangolo non mutano; ai tre lati a_2, b_2, c_2 corrisponderanno nella figura inversa tre circonferenze di cerchio a_1, b_1, c_1, passanti per il punto M, ed i tre centri d_{a_1}, d_{b_1}, d_{c_1} sono situati sulle rette Mt_{a_2}, Mt_{b_2}, Mt_{c_2}; ai punti t_{a_2}, t_{b_2}, t_{c_2} corrisponderanno gli incontri t_{a_1}, t_{b_1}, t_{c_1} delle tre circonferenze a_1, b_1, c_1, colle 3 ultime rette; quelli stavano in linea retta, questi staranno sopra un' altra circonferenza s_1 passante pure per M; ai vertici del triangolo corrisponderanno gli incontri delle tre circonferenze a_1, b_1, c_1; quelli stavano sopra una circonferenza passante per M, questi staranno in linea retta. Il teorema trasformato è dunque il seguente.

Se si hanno quattro circonferenze s_1, a_1, b_1, c_1 passanti per uno stesso punto M, e si prendono gli incontri t_{a_1}, t_{b_1}, t_{c_1} di s_1 con a_1, s_1 con b_1, s_1 con c_1. se questi punti, ed i centri d_{a_1}, d_{b_1}, d_{c_1} sono situati sopra rette uscenti da M. i 3 punti di incontro di a_1, b_1, c_1 sono in linea retta.

Trasformiamo di nuovo questa figura, prendendo la curva polare reciproca rispetto ad un cerchio come conica fondamentale. Sappiamo che la conica polare reciproca di un cerchio rispetto ad un altro cerchio come conica fondamentale (*vedi* Salmon, *A Treatise on Conic Sections*) è una conica di cui un fuoco è il centro del cerchio fondamentale, e la direttrice la polare del centro del cerchio scelto.

Ciò premesso, prendiamo la figura polare reciproca della precedente rispetto ad un cerchio di centro F. Avremo 4 coniche s, a, b, c, aventi un puoco comune F, ed una tangente comune m. Ai punti di incontro del cerchio s cogli a, b, c, corrisponderanno le altre tre rette t_a, t_b, t_c tangenti comuni alle coniche s, a, s, b, s, c; ai centri dei cerchi d_{a_1}, d_{b_1}, d_{c_1} corrisderanno le direttrici d_a, d_b, d_c delle coniche a, b, c: le tre terne di punti $Mt_{a_1}d_{a_1}$, $Mt_{b_1}d_{b_1}$, $Mt_{c_1}d_{c_1}$ erano in linea retta; le tre terne di rette $mt_a d_a$, $mt_b d_b$, $mt_c d_c$ passeranno per un punto.

Ai punti di incontro dei tre cerchi a_1, b_1, c_1 corrisponderanno le tre tangenti alle coniche a, b, c; quei punti erano in linea retta; quelle rette passeranno per un punto. Teorema che è appunto quello che si voleve dimostrare.

[The theorem in the Question is the reciprocal of that given in Ex. 7 of Art. 104 of Salmon's *Conics*, 5th ed.]

5519. (By S. Roberts, M.A.)—**If S and T are the fundamental invariants, and H is the Hessian, of the cubic curve U = 0, prove that the**

twelve lines on which the inflexions lie in threes are represented by

$$S^2U^4 + TU^3H - 18SU^2H^2 - 27H^4 = 0.$$

I. *Solution by* J. J. WALKER, M.A.

In the *Quarterly Journal*, No. 55, October, 1876, p. 243, I have shown that $\lambda U + H$ is the product of xyz, one of the four triads of lines referred to, if λ is a root of the equation

$$27\lambda^4 + 18S\lambda^2 + T\lambda - S^2 = 0.$$

For the continued product

$$(\lambda_1 U + H)(\lambda_2 U + H)(\lambda_3 U + H)(\lambda_4 U + H),$$

where $\lambda_1 \dots \lambda_4$ are the four roots of the above quartic, we have

$$(\lambda_1\lambda_2\lambda_3\lambda_4)\, U^4 + \Sigma(\lambda_1\lambda_2\lambda_3)\,.\,U^3H + \Sigma(\lambda_1\lambda_2)\,.\,U^2H^2 + \Sigma\lambda_1\,.\,UH^3 + H^4,$$

or
$$-\frac{S^2U^4}{27} - \frac{TU^3H}{27} + \frac{18SU^2H^2}{27} + H^4,$$

which, equated with zero, is the equation of the twelve lines in question.

II. *Solution by the* PROPOSER.

Taking the form $\quad x^3 + y^3 + z^3 + 6mxyz = U = 0$ .... (a),
we have, for xyz, $\quad m^2U + H = (1 + 8m^3)\, xyz$................................(b).
But (a) may take the three forms

$$x'^3 + y'^3 + z'^3 + 6\,\frac{1 - \omega m}{1 + 2\omega m}\, x'y'z' = 0,$$

corresponding to the three cube roots ω of unity. Hence, multiplying together the four functions corresponding to (b) and omitting constant multiples, we get for the set of twelve lines

$$(m^2U + H)\left\{[(1 + 2m^3)\, U - 6mH]^3 - 27\,(m^2H + H)^3\right\} = 0 \text{.........} (c).$$

The coefficient of U^4 is

$$m^2(1 - m^3)^2(1 + 8m^3) = S^2(1 + 8m^3),$$

that of U^3H is $\qquad T(1 + 8m^3),$

and so on. Dividing out $1 + 8m^3$, we get the expression given.

(c) is also directly obtained by means of

$$x^3 + y^3 + z^3 - 3\omega xyz = 0,$$

representing three lines containing the nine inflexions of the cubic.

5438. (By the EDITOR.)—If A, B, C, D are four points taken at random on the perimeter of a regular n-gon, find the respective probabilities that AB, CD will intersect (1) inside, (2) on, (3) outside the perimeter.

5506. (By the EDITOR.)—If A, B, C, D, E, F are six points taken at

random in the perimeter of a regular n-gon, find the probability that the three intersections of (AB, CD), (CD, EF), (EF, AB) will all lie inside the perimeter.

Solution by G. S. Carr.

With each configuration of the 4 points there are 3 ways of drawing the pair of chords and therefore 3 intersections.

If the lengths of the sides of the polygon be a, b, c ..., taken at first unequal, the total number of configurations will be represented by

$$(a+b+c+\dots)^4.$$

With each configuration there will be one intersection within the figure, excepting when more than 2 points lie on a side of the polygon, and the number of such unavailable cases will be given by those terms in the expansion of the above multinomial, which are of the forms a^4 and a^3b, that is, $\Sigma a^4 + 4\Sigma a^3 b$.

Putting a, b ... each equal to unity, this number becomes $n+4n(n-1)$, or $4n^2-3n$; therefore the number of interior intersections is n^4-4n^2+3n, and the whole number of intersections is $3n^4$; hence the probability of one interior intersection is $P_1 = \dfrac{n^3-4n+3}{3n^3}$.

For an intersection outside the polygon, more than two points must again not lie on any one side. Therefore, by the last result, the number of exterior intersections would be $2(n^4-4n^2+3n)$, two for each interior intersection. But $6n$ of these occur at the corners of the polygon, arising out of n configurations of two points on one side, and two on the adjacent side multiplied by the number of combinations of the four points, two and two.

Therefore the probability of an exterior intersection is $P_3 = \dfrac{2n^3-8n}{3n^3}$.

The remaining intersections may be considered to fall on the sides of the polygon, and therefore $P_2 = \dfrac{12n-3}{3n^3}$, from $1-P_1-P_3$.

[These results agree with those obtained in Mr. Leudesdorf's solution of the problem; *Reprint*, Vol. XXVIII., pp. 106—108.]

When six points instead of four are taken, there are 15 ways of drawing the 3 chords in each configuration of the six points, but in one only of these ways will the 3 points of intersection lie within the polygon—viz., when opposite points are joined.

The total number of configurations, as in the previous question, will be represented by $(a+b+c+\dots)^6$.

The configurations not available will be those in which 4 or more points lie on the same side of the polygon. The number of these will be expressed by the sum of those terms in the expanded multinomial that involve a fourth or higher power, that is,

$$\Sigma a^6 + 6\Sigma a^5 b + 15\Sigma a^4 b^2 + 30\Sigma a^4 bc.$$

Putting a, b, c, each $=1$, this gives, for the number of restricted cases, $n^6 - n(15n^2-24n+10)$; and the whole number of cases is $15n^6$; therefore the probability of the required event is $\dfrac{n^5-15n^2+24n-10}{15n^5}$.

5470. (By C. W. Merrifield, F.R.S.)—Prove that broken stone, for roads, cannot weigh less per yard than half the weight of a solid yard of the same material, assuming that none of the broken faces are concave, and that it is shaken down so that there shall be no built-up hollows.

Solution by the Proposer.

Observe (1) it cannot lie looser than when all the pieces are the same size and shape; (2) or than when these pieces are regular tetrahedrons; (3) a plane can be completely covered with tetrahedrons, of which the volume will be as one-third the height; (4) we cannot invert one such set on another, so as to fit in, but we can invert a half set so as to fit in with a whole set. This gives $\frac{1}{3}+\frac{1}{6}$ $(=\frac{1}{2})$ of the space filled, and the other half empty. This arrangement, and reasoning, are easily seen to be projective. Hence the theorem.

5304. (By Professor Clifford, F.R.S.)—Prove that the negative pedal of an ellipse, in regard to the centre, has six cusps and four nodes; find their positions, and the length of the arc external to the ellipse between two real cusps; and account fully for the apparent reduction of the curve to a circle and two parabolas respectively, in special cases.

Solution by J. Hammond, M.A.

There are four distinct curves inverse to the conic sections, viz., a bicircular quartic or circular cubic according as the centre of inversion is not or is on the conic, with a node or cusp at the centre of inversion according as the conic is a central conic or parabola. The characteristics of the inverse in these four cases are respectively

m	δ	κ	n	τ	ι
4	3	0	6	4	6
4	2	1	5	2	4
3	1	0	4	0	3
3	0	1	3	0	1

The negative pedal being the polar reciprocal of the inverse, its characteristics are found by interchanging m and n, δ and τ, κ and ι; and it is noticeable that the inverse and negative pedal of the parabola with respect to a point on the curve have the same characteristics.

In the particular case of the central negative pedal of the ellipse, it is at once seen that there are four nodes, six cusps, and three double tangents. When the negative pedal of a given curve is found by first inverting, and then reciprocating, the inverse; the value of the constant of inversion is immaterial provided the constant of reciprocation is equal to it. But when each of these constants is taken to be unity, the projective equation to the inverse becomes the tangential equation to the negative pedal by simply changing the Cartesian coordinates (x, y) into Booth's tangential

coordinates (ξ, υ). Inverting then with a unit radius of inversion the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $(x^2+y^2)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ ....(1),
the projective equation of the inverse, or tangential equation of the negative pedal.

The double tangents of the inverse are clearly parallel to the axes of the ellipse, and are found by giving to either variable such a value as to make equation (1) a perfect square in the other.

Solving (1) for y^2, we get

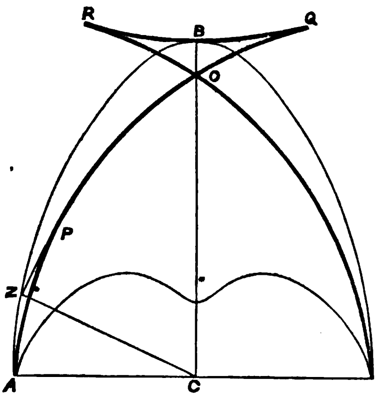
$$2y^2=-2x^2+\frac{1}{b^2}\pm\left\{\left(2x^2-\frac{1}{b^2}\right)^2-4\left(x^4-\frac{x^2}{a^2}\right)\right\} ;$$

and since the quantity under the radical is to vanish, $x^2=\frac{1}{4}e^{-2}b^{-2}$, and y^2 is positive or negative according as $e^2 > < \frac{1}{2}$. There are, therefore, for the negative pedal two real nodes on the axis of x at distances $x=\pm 2eb$, the tangents at the nodes being real or imaginary according as

$$e^2 > < \tfrac{1}{2}.$$

In exactly the same way it is found that the other two nodes lie on the axis of y at distances

$$y=\pm 2iea^2b^{-1}.$$



From the equation to the negative pedal given in BOOTH's *New Geometrical Methods*, Vol. I., p. 143, and in *Reprint*, Vol. XX., p. 108, Question 3431, viz.,

$$\left.\begin{aligned} Q^{12}-(a^2+b^2)^2\,Q^8-18a^2b^2\,(a^2+b^2)\,R^2Q^4 \\ +16\,(a^2+b^2)^3\,a^2b^2R^2+27a^4b^4R^4=0 \end{aligned}\right\} \text{......................(2),}$$

where $Q^4=a^2x^2+b^2y^2+4a^2b^2$, and $R^2=x^2+y^2$, it is easily seen that there are two imaginary cusps at infinity in the directions $a^2x^2+b^2y^2=0$.

The other four cusps are more easily found from the equation

$$P=b\,(1-e^2\sin^2\lambda)^{-\frac{1}{2}},$$

[BOOTH's *New Geometrical Methods*, Vol. I., p. 196, eq. (b)], where P is the perpendicular on the tangent, and λ the angle it makes with the minor axis.

At a cusp $\frac{d^2P}{d\lambda^2}+P$ [see BOOTH's *Methods*, Vol. I., p. 183, eq. (d)] changes sign.

Now $$\frac{dP}{d\lambda}=\frac{be^2\sin\lambda\cos\lambda}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}}=\frac{e^2}{b^2}P^3\sin\lambda\cos\lambda;$$

therefore $$\frac{e^2}{b^2}\cdot\frac{d}{d\lambda}(P^3\sin\lambda\cos\lambda)+P=0$$

gives $$\frac{e^2}{b^2}P^3(\cos^2\lambda-\sin^2\lambda)+\frac{3e^2}{b^2}P^2\sin\lambda\cos\lambda\,\frac{dP}{d\lambda}+P=0.$$

Substituting for $\frac{dP}{d\lambda}$, we have $\frac{e^2(\cos^2\lambda-\sin^2\lambda)}{1-e^2\sin^2\lambda}+\frac{3e^4\sin^2\lambda\cos^2\lambda}{(1-e^2\sin^2\lambda)^2}+1=0$;

therefore $(1-e^2\sin^2\lambda)^2+e^2(1-2\sin^2\lambda)(1-e^2\sin^2\lambda)$
$+3e^4\sin^2\lambda(1-\sin^2\lambda)=0$;

and, finally, $\sin^2\lambda=\frac{1+e^2}{2e^2(2-e^2)}$, $\cos^2\lambda=\frac{(2e^2-1)(1-e^2)}{2e^2(2-e^2)}$ ....(3),

giving four more cusps situated symmetrically one in each quadrant.

The length of the curve is

$$s=\int Pd\lambda+\frac{dP}{d\lambda}=b\int\frac{d\lambda}{(1-e^2\sin^2\lambda)^{\frac{1}{2}}}+\frac{be^2\sin\lambda\cos\lambda}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}};$$

and at the point midway between two cusps $\lambda=\frac{1}{2}\pi$. Taking then double the length from $\lambda=\lambda$ to $\lambda=\frac{1}{2}\pi$, we have for the length of the arc external to the ellipse between two real cusps,

$$s=2b\left\{\int_{\lambda}^{\frac{1}{2}\pi}\frac{d\lambda}{(1-e^2\sin^2\lambda)^{\frac{1}{2}}}-\frac{e^2\sin\lambda\cos\lambda}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}}\right\}\text{ ....(4).}$$

And it is evident from (3) that when $e^2=1$ or $\frac{1}{2}$, $\lambda=\frac{1}{2}\pi$; when $e^2=1$, the second term of (4) becomes infinite; when $e^2=\frac{1}{2}$, $\lambda=\frac{1}{2}\pi$, and the whole expression is zero; and when $e^2<\frac{1}{2}$, all the cusps are imaginary.

When $a=b$, the ellipse reduces to a circle, and it is clear that the negative pedal ought also to be a circle.

In fact, when $b=a$, the tangential equation

$$(\xi^2+v^2)^2=\frac{\xi^2}{a^2}+\frac{v^2}{b^2}\text{ becomes }a^2(\xi^2+v^2)=1.$$

And the projective equation (2) also reduces to that of a circle, for when $b=a$, $Q^4=a^2(R^2+4a^2)$, whence, after reduction and division by a^6R^4, the equation becomes $R^2-a^2=0$. When a is infinite and b finite, the ellipse degenerates into a pair of parallel lines $y^2=b^2$. The tangential equation in this case reduces to a pair of parabolas with a common focus at the origin, as it is clear from geometry that it ought; viz., the equation is $\xi^2+v^2=\pm b^{-1}v$. In this case, dividing the projective equation (2) by a^8, and striking out the terms that now appear with negative powers of a, we easily obtain $(x^2-4b^2)^2=16b^2y^2$, the projective equation of the same pair of parabolas.

Now it has been shown that, when $e^2<\frac{1}{2}$, all the cusps are imaginary, and that there are two acnodes on the axis of x at distances $\pm 2eb$, and no other real singularities.

Thus, as e^2 increases from 0 to $\frac{1}{2}$, the curve is of an oval form, and the acnodes separate from each other till they join on to the curve, when $e^2=\frac{1}{2}$, and consequently $2eb=a$.

For this value of e^2 both the cusps coincide with the node at the extremity of the major axis, forming a triple point with coincident tangents, and after e^2 has passed the value $\frac{1}{2}$ the node and both cusps become real, the distance between the cusps increasing as e increases until, when $e=1$, both cusps at both extremities of the major axis have gone to infinity, and the curve has assumed the form of two parabolas intersecting in the points $(\pm 2b, 0)$, the limiting position of the real node.

The three double tangents corresponding to the three nodes of the inverse are the lines CI, CJ to the circular points, and the line infinity corresponding to the circular points and the origin.

The line infinity is not an ordinary double tangent, but the tangent at each of the infinite cusps.

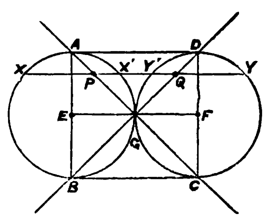
The points of contact of the double tangents CI and CJ are easily found by putting $R^2 = 0$ in (2), whence it is seen that they lie on the imaginary ellipse $Q^4 = 0$, the other points where they cut the curve lying on the ellipse $Q^4 = (a^2+b^2)^2$.

[In the figure, besides the negative pedal, we have inserted its reciprocal, the inverse of the ellipse, in order to show the double tangent and two inflexions corresponding to the node and two cusps of the negative pedal.

Mr. HAMMOND remarks that it would be interesting to investigate the forms and properties of some other negative pedals. Of these one of the simplest is that of the parabola, with the origin at the vertex, where the inverse is a Cissoid and the negative pedal a Semicubical Parabola. The negative *focal* pedal of the ellipse has been investigated in the *Reprints* (Vols. XVI., pp. 77—83; XVII., pp. 92—96; XX., pp. 106, 107), under Quests. 3430, 3550; and the form of the curve—which resembles that of the *central* pedal given above—is shown in the diagram on p. 77 of Vol. XVI. The central pedal is discussed in Vols. XVI., pp. 83—85, and XX., pp. 107, 108, under Quest. 3431. It was, however, inadvertently stated (XVI., p. 84, Art. 2) that "the curve lies wholly inside the ellipse"; whereas, by eq. (5) on p. 84, it is at once seen that $y = 0$ when $\cos\psi = \dfrac{b}{ae} = \dfrac{b}{c} = \dfrac{(1-e^2)^{\frac{1}{2}}}{e}$, which is possible when $e^2 > \frac{1}{2}$; and that then by eq. (4) the pedal crosses the axis at O where $CO = x = 2eb$, as found above. The form of the curve may be easily traced from eqs. (1) to (7), as done for the focal pedal in Art. (8) on p. 80 of Vol. XVI. The excentric angle ψ that corresponds to the node O is constructed (as in the figure to Quest. 3656, *Reprint*, Vol. XXVII., p. 96) by drawing to the circle on the minor axis a tangent from the focus, which, since $e^2 > \frac{1}{2}$, must in this case lie outside the circle.]

5479. (By A. W. PANTON, M.A.)—1. If circles be drawn on a pair of opposite sides of a square; prove that (1) the polars of any point on either diagonal with respect to the two circles meet on the other diagonal, and (2) the four tangents from the point form an harmonic pencil.

I. *Solution by* Professor TOWNSEND, F.R.S.



Let ABCD be the square, G the intersection of its diagonals, E and F the centres of the two circles, PQ any chord parallel to EF of the angle determined by the diagonals, and XX′ and YY′ the two chords of intersection, real or imaginary, of PQ with the circles; then, since XX′ and YY′ are manifestly both cut harmonically by PQ, therefore the polars of P with respect to both circles pass through Q, and conversely; and therefore &c. as regards the first part of the property.

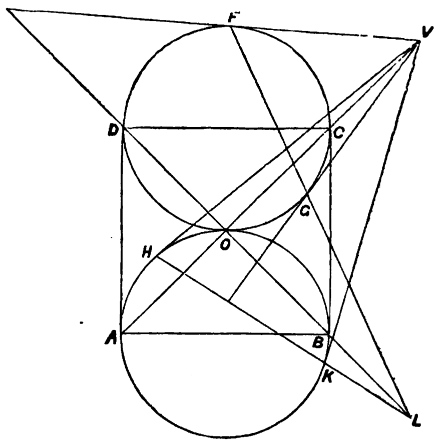
The second is manifestly but a particular case of the more general and well-known property, that the locus of points, whose angles of subtense of any two conics cut each other harmonically, is the conic passing through the eight points of contact of their four common tangents, which, in the case of the two circles in question, manifestly breaks up into the two diagonals of the square; but it may be proved directly for the particular case as follows:—Denoting by α and β the two angles of subtense, real or imaginary, of the circles from P or Q; by γ that of the connector EF of their centres from the same; by e, f, and g the three distances of P or Q from E, F, and G; and by $2a$ the distance EF; the property will manifestly be proved, if it be shown that $\cos\alpha \,.\, \cos\beta = \cos 2\gamma$, or that

$$2\left(\frac{e^2+f^2-4a^2}{2ef}\right)^2 - 1 = \left(1-2\frac{a^2}{e^2}\right)\left(1-2\frac{a^2}{f^2}\right),$$

or that $$(e^2+f^2-4a^2)^2 - 2e^2f^2 = 2(e^2-2a^2)(f^2-2a^2),$$

which, as $e^2 = a^2+g^2 \mp \sqrt{2}\,.\,ag$, and $f^2 = a^2+g^2 \pm \sqrt{2}\,.\,ag$, is manifestly the case.

II. *Solution by* R. Tucker, M.A.; E. Rutter; *and others.*



Refer the circles to OB, OC as axes, and take OV = K, AB $= a\sqrt{2}$; then the equations are

$$x^2+y^2+ax-ay = 0, \quad x^2+y^2-ax+ay = 0 \text{ .... } (1, 2).$$

The polars of V are, for (1) and (2) respectively,

$$2yk+ax-ay-ak = 0, \quad 2yk-ax+ay+ak = 0 \text{ .... } (3, 4);$$

that is, each meets OB in L, so that OL = K = OV.

Now transform to parallel axes through V, and our equations are

$$x^2+(y+k)^2+ax-a(y+k) = 0, \quad x^2+(y+k)^2-ax+a(y+k) = 0 \ldots (5, 6).$$

Combine with these equations $y = mx$, and introduce the condition for

tangency, and we have

$$a^2m^2 - 2a(a-2k)m + a^2 + 4ak - 4k^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (7),$$
$$a^2m^2 - 2a(a+2k)m + a^2 - 4ak - 4k^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

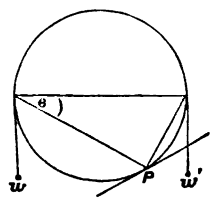
Now if $\quad y = m_1x, \quad y = m_2x, \quad y = m'_1x, \quad y = m'_2x,$
be the four tangents, these form an harmonic pencil, if

$$2m_1m_2 + 2m'_1m'_2 = (m_1+m_2)(m'_1+m'_2).$$

From (5) and (6) this condition is seen to be fulfilled, hence &c.

4382. (Proposed by F. C. Wace, M.A.)—At the extremities of the horizontal diameter of a circular wire are fixed two small rings, a third ring can slide on the wire, a string passes through the two rings and supports two weights w, w' hanging vertically; find the position of the moveable ring when it is in equilibrium.

Solution by the Rev. T. J. Sanderson, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



Supposing *two* strings extended, each attached to the sliding ring, and neglecting friction, we have, for equilibrium, resolving along the tangent at P, $\qquad w'\cos\theta = w\sin\theta,$

therefore $\qquad \tan\theta = \dfrac{w'}{w},$

which gives the position of P.

5563. (By Professor Sylvester, F.R.S.)—If at two points in a cubic curve, lying in a straight line with a point of inflexion, tangents be drawn to meet the curve again, prove that their intersections with it will also be in a straight line with the same point of inflexion.

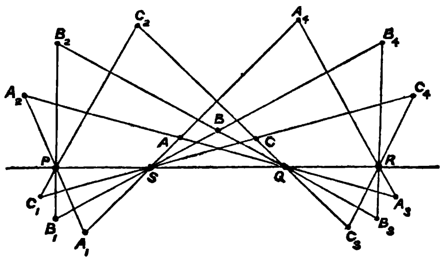
Solution by J. Hammond, M.A.

This elegant theorem, like Professor Sylvester's construction for drawing a tangent to a cubic (Quest. 4869, *Reprint*, Vols. XXV., p. 27, and XXVI., p. 32), follows at once from one given in Salmon's *Higher Plane Curves* (2nd ed., Art. 149), viz.,—The tangents at any collinear triad of points on a cubic meet the curve again in a collinear triad of points. For the inflexional tangent meeting the curve again at its point of contact, the other tangents meet the curve again in points in a line with the point of inflexion.

5575. (By the EDITOR.) — If of four triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, the first is in perspective with the second, the second with third, the third with fourth, and the fourth with first, in such a way that the vertices of the same letters are corresponding; and if the four centres of perspective lie in a straight line: prove that the four perspective lines meet in a point.

Solution by Prof. TOWNSEND, F.R.S.

The three triangles A_1A_2A, B_1B_2B, C_1C_2C, taken in pairs, having a common axis of perspective PQS (see the Figure, which explains itself), so therefore, by a well known property of triangles in perspective, have the three $A_1B_1C_1$, $A_2B_2C_2$, and ABC; and the three triangles A_3A_4A, B_3B_4B, C_3C_4C, taken in pairs, having a common axis of perspective QRS, so therefore, by the same property, have the three $A_3B_3C_3$, $A_4B_4C_4$, and ABC; consequently, the axis of perspective of the pairs $A_1B_1C_1$ and $A_2B_2C_2$ coincides with that of either of them and ABC, and the axis of perspective of the pair $A_3B_3C_3$ and $A_4B_4C_4$ coincides with that of either of them and ABC. Again, the three triangles $A_2B_2C_2$, $A_3B_3C_3$, and ABC, taken in pairs, having a common centre of perspective Q, their three axes of perspective, by the reciprocal property, are concurrent, and therefore, by the above, so are those of the three pairs of triangles $A_1B_1C_1$ and $A_2B_2C_2$, $A_2B_2C_2$ and $A_3B_3C_3$, $A_3B_3C_3$ and $A_4B_4C_4$; and the three triangles $A_4B_4C_4$, $A_1B_1C_1$, and ABC, taken in pairs, having a common centre of perspective S, their three axes of perspective, by the same property, are concurrent; and therefore, by the above, so are those of the three pairs of triangles $A_2B_2C_2$ and $A_3B_3C_3$, $A_3B_3C_3$ and $A_4B_4C_4$, $A_4B_4C_4$ and $A_1B_1C_1$; therefore &c.



N.B.—The converse of the above property is manifestly the reciprocal of the original, and may consequently be regarded as established with it by reciprocation, either of the property itself or of the process employed for its establishment as above.

Postscript.—The above demonstration, based essentially on the supposition, since seen to be unnecessary, that the planes of the three quadrilaterals $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$, containing as they always do the common line PQRS, were coincident, may be simplified for the more general case when they are not, as follows:—The three triangles A_1A_2A, B_1B_2B, C_1C_2C, taken in pairs, having a common axis of perspective PQS, the two planes $A_1B_1C_1$ and $A_2B_2C_2$ consequently, by the property above referred to, intersect in the plane ABC; and the three triangles A_3A_4A, B_3B_4B, C_3C_4C, taken in pairs, having a common axis of perspective QRS, the two planes $A_3B_3C_3$ and $A_4B_4C_4$ consequently, by the same well-known property, intersect in the plane ABC; therefore the four planes $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$ intersect at a common point O in the plane ABC; and therefore &c.

N.B.—If A′, B′, C′ be the triad of sixth vertices of the three complete quadrilaterals $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ in the above, it appears at once from the demonstration just given that the point O lies also in the plane A′B′C′, and therefore that the six planes $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, ABC, and A′B′C′ are concurrent.

5586. (By R. E. Riley, B.A.)—If a series of circles be described concentric with an ellipse of eccentricity e, prove that the chords of contact of tangents drawn to the ellipse from points in these circles envelop a series of concentric similar ellipses of eccentricity $e\,(2-e^2)^{\frac{1}{2}}$.

Solution by the Editor.

Let $$x^2+y^2 = r^2 \text{ and } a^2\xi^2+b^2v^2 = 1 \qquad\qquad (1,\,2)$$
be respectively the *projective* equation of one of the concentric circles and the *tangential* equation of the given ellipse whose eccentricity is e.

Assume, on the circumference of the circle, a point, as *pole*, whose *projective* coordinates are (x, y); and let (ξ, v) be the *tangential* coordinates of its *polar*, that is to say, of the chord of the ellipse. Then, as in Booth's *Geometrical Methods*, Vol. I., Art. 31, we shall have, from (2), $x = a^2\xi$, $y = b^2v$; hence, substituting these values of x and y in (1), the tangential equation of the envelop required is found to be $a^4\xi^2+b^4v^2 = r^2$, which designates an ellipse whose eccentricity is $e\,(2-e^2)^{\frac{1}{2}}$.

If the pole (x, y) move along the straight line $\dfrac{x}{h}+\dfrac{y}{k} = 1$, a like substitution would give us, for the envelop of the polar, the tangential equation $\dfrac{a^2}{h}\xi+\dfrac{b^2}{k}v = 1$, which (Booth's *Methods*, Vol. I., p. 1) designates a point whose projective coordinates are $\dfrac{a^2}{h}$ and $\dfrac{b^2}{k}$, that is to say, third proportionals to (h, a) and to (k, b).

5478. (By C. B. S. Cavallin.)—Three straight lines are drawn at random across a triangle; show that the probability that each line cuts unequal pairs of sides is $16\,(a+b+c)^{-4}\Delta^2$, where Δ is the area of the triangle and a, b, c its sides.

Solution by E. B. Elliott, M.A.

Following Professor Crofton's theory, the number of lines crossing the triangle is measured by the perimeter $a+b+c$. Consequently the measure of the number of sets of three lines crossing it is $(a+b+c)^3$.

Again, $2a$ measures the number of lines which meet the side a of the triangle. Therefore the measure of the number meeting b and c, that is to say, of all the lines which cross the triangle except those which meet a, is $a+b+c-2a = b+c-a$. So also $c+a-b$ and $a+b-c$ measure the numbers of lines meeting c, a and a, b, respectively. Thus

$$(b+c-a)(c+a-b)(a+b-c)$$

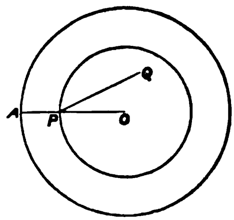
measures the number of sets of three lines, one meeting b, c, a second c, a, and the third a, b. Hence the required probability is

$$\frac{(b+c-a)(c+a-b)(a+b-c)}{(a+b+c)^3} = \frac{16\Delta^2}{(a+b+c)^4}.$$

5496. (By Professor Crofton, F.R.S.)—Prove that the mean value of the reciprocal of the distance of any two points within a circle of radius r is

$$M\left(\frac{1}{\rho}\right) = \frac{16}{3\pi r}.$$

Solution by E. B. Seitz.



Let P and Q be any two points within the circle, centre O, and while P ranges over the circle, let Q be confined to the concentric circle whose circumference passes through P.

Let $OA=r$, $PQ=x$, $PQ=y$, $\angle OPQ=\theta$; then an element of the circle at P is $2\pi x\,dx$, and at Q it is $d\theta\, y\,dy$; the limits of x are 0 and r, those of θ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and of y, 0 and $2x\cos\theta$. Hence, doubling, since P may be confined to the concentric circle whose circumference passes through Q, we have

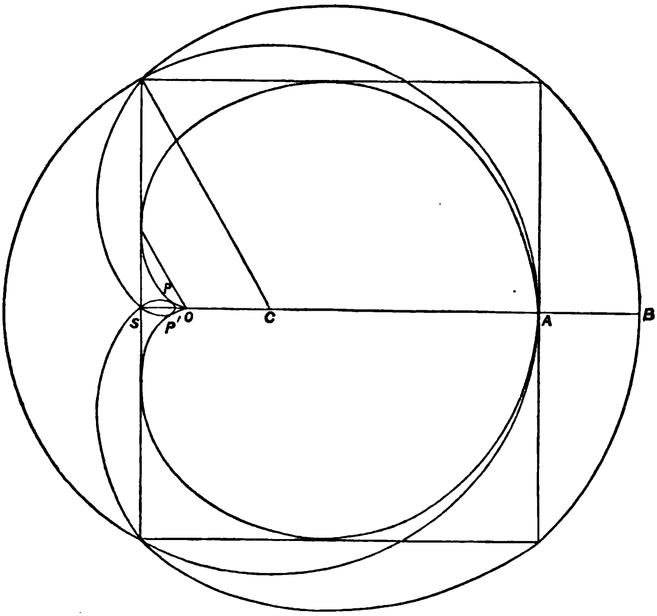
$$M\left(\frac{1}{\rho}\right) = \frac{2}{\pi^2 r^4}\int_0^r \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{2x\cos\theta} \left(\frac{1}{y}\right) 2\pi x\,dx\,d\theta\,y\,dy$$

$$= \frac{8}{\pi r^4}\int_0^r \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x^2 dx \cos\theta\, d\theta = \frac{16}{\pi r^4}\int_0^r x^2\,dx = \frac{16}{3\pi r}.$$

5458. (By Professor Wolstenholme, M.A.)—Find the locus of the intersection of perpendicular tangents to a cardioid, and trace the resulting curve.

Solution by R. TUCKER, M.A.

In the *Proceedings* of the London Mathematical Society (Vol. IV., pp. 327—330), Professor WOLSTENHOLME has discussed this locus and drawn the figure. The locus is made up of a circle and a bicircular quartic.



[Prof. WOLSTENHOLME sent the accompanying figure, with the question, long before the above-cited article appeared in the Mathematical Society's *Proceedings*. In a note that accompanied the figure, it was stated that the locus of the intersection of tangents, inclined at a constant angle to each other, to any epi- or hypo-cycloid is a trochoid, a more general form whereof this is a particular case. In regard to the above form of the locus, it was added that it is rather too much flattened near the vertex; AB should be a little greater than OC; $SB = 3(2+\sqrt{3})a$, $OB = (5+3\sqrt{3})a$, which is a little greater than $10a$, $= 10\cdot 2a$ nearly, therefore $AB = 2\cdot 2a = 1\cdot 1$ OC. The Proposer stated, furthermore, that in all cases in which he had seen perpendicular tangents to a cardioid mentioned, he had only noticed the circular locus taken account of.]

5146. (By S. ROBERTS, M.A.)—Given a pencil of rays and a system of concentric circles; prove (1) that if one set of intersections range on a

straight line, the other intersections lie on a circular cubic, having a double point at the origin of the pencil and the double focus at the common centre of the circles; and (2) determine therefrom, with reference to a system of parabolas having the same focus and axis, the locus of the points the normals at which intersect in a fixed point.

I. *Solution by* E. B. Elliott, M.A.; Prof. Evans, M.A.; *and others.*

1. More generally, take any system of similar, similarly situate, and concentric conics $u_2 + u_1 + c = 0$ ....(1), c being arbitrary; and let the origin be the vertex of a pencil of lines.

Let one set of intersections range on the straight line $v_1 = 1$. Required the locus of the other set of intersections.

The equation $u_2 + v_1 u_1 + c v_1^2 = 0$

is that of the rays of the pencil corresponding to the conic (1). Eliminating, then, c between this and (1), the equation

$$u_2 + v_1 u_1 - (u_2 + u_1)\, v_1^2 = 0$$

is found as that of the locus of intersections of the pencil and system. This is, of course, divisible by $v_1 - 1$, and there results

$$u_2 v_1 + u_1 v_1 + u_2 = 0$$

as the equation of the locus of the second intersections. It is a cubic having a double point at the origin, having $v_1 + 1 = 0$ for one asymptote, and for the other two the asymptotes of $u_2 + u_1 = 0$, that is to say, the common asymptotes of the system (1).

In particular, then, if the system (1) be one of concentric circles, the locus is a cubic having the asymptotes of that system for a pair of asymptotes; that is to say, a circular cubic with its double focus at the centre of the system, and having the origin, the vertex of the pencil, for a double point.

2. A circle whose centre is at the focus of a parabola, and which passes through any point on the curve, passes also through the point where the normal at that point meets the axis.

Thus, drawing normals to the system of parabolas from any point, and describing circles with the focus as centre to pass through all their feet, we have a pencil and a system of circles as above, the straight line which is the locus of one intersection being the axis.

Thus the locus of the feet of the normals, the second set of intersections, is a circular cubic having a double point at the point from which normals are drawn, and the focus of the system of parabolas for its double focus. Its third asymptote is a line parallel to the axis, and at the same distance on one side of the fixed point as the axis is on the other.

II. *Solution by* C. Leudesdorf, M.A.; Prof. Wolstenholme; *and others.*

(1). Take the vertex of the pencil as origin, and let the equation of one of the circles be $r^2 - 2dr\cos\theta + d^2 - \rho^2 = 0$...........................(1); and let the fixed straight line be

$$a\cos\theta + b\sin\theta + \frac{c}{r} = 0 \text{..........................(2).}$$

Now, if r_1, r_2 be the roots of (1), we have $r_1+r_2 = 2d \cos\theta$;
and since (r, θ) satisfies (2), we have

$$(2d \cos\theta - r_1)(a \cos\theta + b \sin\theta) + n = 0;$$

or, in rectangular coordinates,

$$(x^2+y^2-2dx)(ax+by)+c = 0,$$

a circular cubic, having a double point at the origin, and its double focus at the point $(d, 0)$, the centre of the circles.

(2). Calling the fixed point N, draw a normal PNG cutting the axis in G; then, if S be the focus, a circle centre S and radius SP will pass through G; and, considering the series of concentric circles formed by taking successive positions of PNG, we see that the points G range along a fixed straight line, and that therefore, by (1), the locus of P is a circular cubic having a double point at N and the double focus at S.

5502. (By W. S. B. Woolhouse, F.R.A.S.)—(1) Two chords are drawn in a circle; all that is known, appertaining to them, being that they intersect within the circle; determine the respective probabilities that a third random chord shall intersect neither, only one, or both of them within the circle. (2) Two chords are drawn in a circle; all that is known, appertaining to them, being that they do not intersect within the circle; determine the respective probabilities before stated.

I. *Solution by* Professor Henry Stanley Monck, M.A.

1. Let the ends of the two chords be numbered in order (*i.e.*, in the same direction) 1, 2, 3, 4. Since they intersect, the chords must be formed by joining 1, 3 and 2, 4. We now draw a third chord. Let one of its ends fall between 1 and 2 (by properly selecting one starting point we can always provide that this shall be so). Then the other extremity being taken altogether at random, and the points 1, 2, 3, 4 being random points, the chances are equal that it lies between 1 and 2, between 2 and 3, between 3 and 4, and between 4 and 1. In the first of these cases the third chord will not intersect either of the former, in the second and fourth it will intersect one of them, in the third both. Hence the chances are that it will intersect both $\frac{1}{4}$, that it will intersect neither $\frac{1}{4}$, that it will intersect one only $\frac{1}{2}$.

This differs from Miss Blackwood's result (in Quest. 5461, see *Reprint*, Vol. XXVIII., p. 109), as, taking the chance of two chords drawn at random intersecting as $\frac{1}{3}$, the chance of three intersecting each other would be $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ instead of $\frac{1}{15}$. For brevity, I speak of intersection within the circle as intersection simply.

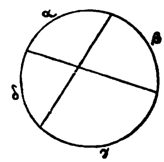
2. Since the two chords do not intersect, 1 must be connected with 2, and 3 with 4, or else 1 with 4, and 2 with 3. It will be enough to consider the former, as the points are arbitrary. Let one extremity of the new chord lie between 1 and 2, as before. The chances are again equal that the other will lie between 1 and 2, 2 and 3, 3 and 4, and 4 and 1, and the chances will be the same as in the former case. But it is now as probable

that the first extremity of the new chord will lie between 2 and 3 as between 1 and 2; and in the latter case if the second extremity lies between 1 and 2 or 3 and 4, there will be one intersection, while, if it lies between 2 and 3 or 1 and 4, there will be none. Taking the whole eight cases together, therefore, the new chord will only intersect both the former chords in one, it will intersect one only in four, and none in three. The chances therefore in this instance are still $\frac{1}{2}$ for a single intersection, $\frac{1}{8}$ for intersecting both, and $\frac{3}{8}$ for intersecting neither.

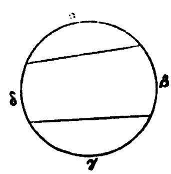
[Mr. WOOLHOUSE remarks that "this solution appears plausible, but is not correct. It would be true if 1, 2, 3, 4 were fixed and equidistant. But when 1, 2, 3, 4 vary, it is fallacious." On communicating this remark to Professor MONCK, he sent thereon the following comments:—"I certainly do not assume the four points to be fixed points, and cannot see where the error in my solution lies. The whole dispute seems to me to turn on whether certain chords ought to be counted twice over or once only. Something may be said for Mr. WOOLHOUSE's way of counting (which is really counting the chord *once* when both extremities are in the same arc, and *twice* when they are in different arcs), but he has not said it. Suppose for example the question were, what is the chance that a random chord will intersect a fixed diameter, is it not clearly $\frac{1}{2}$? Yet this can only be reached, I apprehend, by counting the chords which intersect the diameter twice, and those which do not intersect it once only." See also a further Note (135) on p. 62 of this volume.]

II. *Solution by the* PROPOSER.

Let $\alpha, \beta, \gamma, \delta$ denote the four arcs into which the circumference of the circle is divided by the two chords, as shown for the two cases of the problem in the the annexed diagrams. Also let P, P′ designate the random points connected by the new chord. On each arc the number of possible positions of a random point may be estimated by its length. This being premised, the total number of positions of P, P′ are each 2π, and the total number of chords, reckoning two, viz., PP′, P′P, for each pair of points, is



$$
\begin{aligned}
(2\pi)^2 &= (\alpha+\beta+\gamma+\delta)^2 \\
&= \alpha^2+\beta^2+\gamma^2+\delta^2+2\alpha\beta+2\alpha\gamma+2\alpha\delta \\
&\qquad +2\beta\gamma+2\beta\delta+2\gamma\delta \ldots\ldots\ldots (1).
\end{aligned}
$$



This development, in its several terms, exhibits an analysis of the corresponding classes of chords. Thus, α^2 expresses the number of chords having P, P′ both on the arc α; $2\alpha\beta$ expresses the number of chords having the points in the arcs α and β, viz., P in α, P′ in β, and P′ in α, P in β. The magnitudes of the arcs $\alpha, \beta, \gamma, \delta$ being unknown, it becomes requisite to effect a summation of the number of positions for every possible set of values, the same being known to be equally probable. For this purpose the limits of integration must be such as to exclude negative values, α, β, γ, or δ. Let

$$
\begin{aligned}
L &= \alpha+\beta \\
M &= \alpha+\beta+\gamma \\
2\pi &= \alpha+\beta+\gamma+\delta.
\end{aligned}
$$

Then, taking α, L, M, successively, as variables, we shall have the following limits, viz., α from 0 to L, L from 0 to M, and M from 0 to 2π. Hence, for the total number of chords, we obtain, remembering the limits just stated,

$$\iiint (2\pi)^2\, d\alpha\, dL\, dM = (2\pi)^2 \iint L\, dL\, dM = \frac{(2\pi)^2}{2}\int M^2 dM = \frac{(2\pi)^5}{2\,.\,3} \ldots (2).$$

For the number (α^2) of chords contained by the arc α,

$$\iiint \alpha^2\, d\alpha\, dL\, dM = \tfrac{1}{3}\iint L^3\, dL\, dM = \frac{1}{3\,.\,4}\int M^4 dM = \frac{(2\pi)^5}{3\,.\,4\,.\,5} \ldots\ldots (3).$$

Also, for the number ($2\alpha\beta$) of chords connecting the arcs α and β,

$$\iiint 2\alpha\beta\, d\alpha\, dL\, dM = \iiint 2\alpha\,(L-\alpha)\, d\alpha\, dL\, dM$$

$$= \tfrac{1}{3}\iint L\; dL\, dM = \frac{1}{3\,.\,4}\int M^4 dM = \frac{(2\pi)^5}{3\,.\,4\,.\,5} \ldots\ldots\ldots (4).$$

The probability of the chord PP′ falling wholly on the arc α is therefore $\frac{(3)}{(2)} = \frac{1}{10}$; and the probability of its falling upon the arcs α and β is $\frac{(4)}{(2)} = \frac{1}{10}$. Similarly the probability of PP′ falling upon any one, or upon any specified two, of the arcs $\alpha, \beta, \gamma, \delta$ is $\frac{1}{10}$; and thus the combinations severally represented by the ten terms of (1) have an equal probability. Now, referring to the diagram showing the first case of the problem, we perceive by inspection that the combinations are

for no intersection, $\alpha^2, \beta^2, \gamma^2, \delta^2$;
for one intersection, $2\alpha\beta, 2\beta\gamma, 2\gamma\delta, 2\alpha\delta$;
for two intersections, $2\alpha\gamma, 2\beta\delta$;

the respective probabilities of which are therefore $\frac{4}{10}, \frac{4}{10}, \frac{2}{10}$, that is $\frac{2}{5}, \frac{2}{5}, \frac{1}{5}$,

Again, turning to the second diagram, the combinations are

for no intersection, $\alpha^2, \beta^2, \gamma^2, \delta^2, 2\beta\delta$;
for one intersection, $2\alpha\beta, 2\beta\gamma, 2\gamma\delta, 2\alpha\delta$;
for two intersections, $2\alpha\gamma$;

the respective probabilities of which are therefore $\frac{5}{10}, \frac{4}{10}, \frac{1}{10}$, that is, $\frac{1}{2}, \frac{2}{5}, \frac{1}{10}$.

Note.—Estimating the total intersections of the *three chords* in the two cases of the problem, the results obtained show that the respective probabilities are as stated hereunder:—

	Intersections. 0	1	2	3
Case (1)...........................	0	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
Case (2)...........................	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$	0

In the problem of three random chords, before any chord is drawn, it is known that the chance of Case (1) occurring is $\frac{1}{3}$, and that of Case (2) is $\frac{2}{3}$. Therefore, multiplying the two sets of values just stated respectively by $\frac{1}{3}$ and $\frac{2}{3}$, and adding the products, we find that for three random chords the probabilities of 0, 1, 2, 3 intersections within the circle are $\frac{1}{3}, \frac{2}{5}, \frac{1}{5}, \frac{1}{15}$, respectively, the same as determined in an example contained in my general investigation of those problems for n chords. See my solution to Quest. 1894, *Reprint*, Vol. V., p. 110. It will be seen that the probability of

three intersections agrees with Miss BLACKWOOD's solution to Quest. 5461, (*Reprint*, Vol. XXVIII., p. 109), the notes appended to which gave rise to the recent discussion on the subject of "random chords."

The problem may be solved without the aid of the integral calculus in the following manner:—

Let any six points taken in order round the circumference of the circle be numbered 1, 2, 3, 4, 5, 6. Then, beginning from the first point, it is easy, from an inspection of the diagram, to make up the annexed tables for the two cases. In that relating to Case (1), the first column contains every pair of chords that intersect; the second column contains the third chord, or that connecting the two remaining points; and the last column shows the number of intersections made by the third chord. The table appertaining to Case (2) is made up in like manner, only the first column here contains every pair of chords that do not intersect. For the system of six points, as regards Case (1) we have ten combinations, viz., four with no intersection, four with one intersection, and two with two intersections, the respective probabilities of which are therefore $\frac{4}{10}$, $\frac{4}{10}$, $\frac{2}{10}$, or $\frac{2}{5}$, $\frac{2}{5}$, $\frac{1}{5}$. In Case (2) there are twenty combinations, viz. ten with no intersection, eight with one intersection, and two with two intersections, the respective probabilities of which are therefore $\frac{10}{20}$, $\frac{8}{20}$, $\frac{2}{20}$, or $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{10}$, as before found.

CASE (1).		
Chords intersecting.	Third chord.	New intersections
13, 24	56	0
„ 25	46	1
„ 26	45	0
14, 25	36	2
„ 26	35	1
„ 35	26	1
„ 36	25	2
15, 26	34	0
„ 36	24	1
„ 46	23	0

CASE (2).		
Chords not intersecting.	Third chord.	New intersections
12, 34	56	0
„ 35	46	1
„ 36	45	0
„ 45	36	0
„ 46	35	1
„ 56	34	0
13, 45	26	1
„ 46	25	2
„ 56	24	1
14, 23	56	0
„ 56	23	0
15, 23	46	1
„ 24	36	2
„ 34	26	1
16, 23	45	0
„ 24	35	1
„ 25	34	0
„ 34	25	0
„ 35	24	1
„ 45	23	0

As Case (1) exhibits ten combinations, and Case (2) exhibits twenty combinations, the respective probabilities of their promiscuous occurrence are $\frac{1}{3}$ and $\frac{2}{3}$.

The argument is completed by conceiving the six points to vary so as to severally occupy every position on the circumference. It is evident that every possible formation of diagram must then result for the two cases; and as the combinations arising from each separate system of six points give the foregoing probabilities, it is evident that those probabilities will remain unaltered when all the combinations are included in one total.

III. *Solution by* ELIZABETH BLACKWOOD.

Assuming that a *random chord* means the line joining random points on the circumference, let the random points taken in order be A, B, A′, B′, so that the intersecting chords are AA′ and BB′. The circumference is thus divided into 4 random parts, namely the arcs AB, BA′, A′B′, B′A; the

point C is equally likely to fall on any one of these, and the respective chances will remain the same if we restrict C to *one* of them, say to the arc AB.

Let p_1, p_2, p_3 be the respective chances required. Of the four random arcs into which the circumference is divided the only favourable position of C′ for the chance p_1 is the arc AB. Hence $p_1 = \frac{1}{4}$. There are two favourable positions of C′ for the chance p_2, namely, the arcs BA′ and B′A. Hence $p_2 = \frac{2}{4}$. The only favourable position of C′ for the chance p_3 is the arc A′B′. Hence $p_3 = \frac{1}{4}$. Thus $p_1, p_2, p_3 = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

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135. Note on Mr. Woolhouse's Solution of his Question 5502 (See pp. 59—61 of this volume). *By* Professor Monck, M.A.

The chance that a random point P will be in the arc α is undoubtedly $\frac{\alpha}{\alpha+\beta+\gamma+\delta}$, and the chance that a second random point P′ will also lie in the same arc being identical, the chance that the chord PP′ will lie in it is $\frac{\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$. But if we are to count P′P as a distinct chord from PP′, the chance that P′P will be in the arc α is also $\frac{\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$, and therefore the chance that our geometrical chord (which includes both PP′ and P′P) will lie in the arc α is not $\frac{\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$, but $\frac{2\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$. On the other hand, if we are to reckon PP′ and P′P as a single chord, the chance that one extremity will lie in the arc α, and the other in the arc β, is not $\frac{2\alpha\beta}{(\alpha+\beta+\gamma+\delta)^2}$ but $\frac{\alpha\beta}{(\alpha+\beta+\gamma+\delta)^2}$.

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136. Miss Blackwood's Reply to Helen Thomson's Verses on "Random Chords." (See p. 40 of this volume.)

With arrows fashioned by the Muse's hand
Your Roderick's foe *you* venture to withstand?
So be it, Helen; be it even so;
Another string is fastened to *my* bow;
Another arrow fits unto this string,
Expressly sharpened for your Muse's wing.
But ere my bow-string's awful twang you hear
I'd like to ask one question, Helen dear.
Whence *comes* your "attribute of subject *chord*"?
To trace this attribute let's trace the *word*.
The archer and the minstrel both require
(For trusty bow, or spirit-stirring lyre)
Their "chords" *well fastened at the ends* and tight,—
Else, sure, nor shaft nor music would go right.
My chord is tight, my arrow right will go—
No sluggish shaft, as you will quickly know.
All proper chords are *finite—finite*, mark;
Each chord is finite, *fastened to its arc*.
The *arc* comes *first*, and then the *chord* comes *next*;

A chord without an arc could not exist;
An arc can well exist without a chord,
As common usage has defined this word.
The arc once fixed, we fix the chord as well;
The arc unknown, no soul the chord can tell.
Now points my arrow to its fated mark;
Two random points will give a random arc;
A random arc a random chord implies,
Twang! Whizz! your Muse falls headlong from the skies.

[Fearing, apparently, that the foregoing reply—made under the shackles of verse—is not altogether satisfactory, Miss BLACKWOOD, in a subsequent communication, breaks down into good honest prose, and puts the question at issue in a far better form as follows:—

That the difference between myself and my critics may be reduced within the narrowest possible limits, I may go so far as to admit that *their chords are random lines;* but I cannot admit the inference which they draw therefrom, namely, that *their chords are random chords.* To take an exactly parallel case; I may admit, and do fully and freely admit, that *my critics* are good mathematicians; must I therefore admit as a necessary consequence that my critics are good *critics?* I by no means think I am bound to admit this as a logical inference, though I admit it unhesitatingly as an actual fact—with the qualification, however, that in this particular question of random chords their criticism is altogether at fault. Put into a strict syllogistic form, the argument of each of my opponents is this;—

All my-lines are random; all my-chords are my-lines;
Therefore, all my-chords are random.

To all appearance, the syllogism is perfect; where then is the fallacy? The fallacy arises from the fact that the word *random* is used in the major premiss as an abbreviation for random *lines*, and in the conclusion as an abbreviation for random *chords*,—two phrases which to my mind convey quite different meanings.

By her conversion of the second premiss in the above syllogism, Miss THOMSON would seem also open to the charge of falling into another fallacy, namely, that which logicians term "Illicit Process." But as this conversion is probably a mere verbal sacrifice made to Apollo in order to obtain from him a "chord" for her logical lyre, it would be unfair and ungenerous to press the charge. I know by sad experience that the musical god cannot be propitiated without heavy sacrifices of this kind.

We shall be glad if our correspondents will henceforth stick to prose, since, for subjects such as these, verse is wholly unsuitable.]

5090. (By C. LEUDESDORF, M.A.)—Evaluate (1) the equation

$$(ax^2+by^2+c+2fy+2gx+2hxy)(ax'^2+by'^2+c+2fy'+2gx'+2hx'y')$$
$$=[(ax'+hy'+g)x+(hx'+by'+f)y+(gx'+fy'+c)]^2,$$

when $lx'+my'=0$, and x' and y' become infinite; and (2) give the geometrical interpretation.

I. *Solution by* H. T. GERRANS, B.A.; C. BICKERDIKE; *and others.*

1. Substituting for $x' : y'$ in the given equation, and neglecting terms lower than the second degree in x'^2 and y'^2, we have

$$(x^2l^2 + y^2m^2 + 2xylm)(ab - h^2) + 2x(bgl^2 - ghlm + aflm - b^2fh)$$
$$+ 2y(afm^2 + bglm - fhlm - ghm^2) + c(am^2 + bl^2 - 2hlm) = 0.$$

Suppose this is identical with $(ab - h^2)(lx + my + \alpha)(lx + my + \beta) = 0$; then, equating coefficients, we have

$$(\alpha + \beta)(ab - h^2) = 2(afm + bgl - flh - ghm), \quad \alpha\beta(ab - h^2) = am^2 + bl^2 - 2hlm.$$

Hence the equation becomes $(lx + my + \alpha)(lx + my + \beta) = 0$, that is to say, two parallel straight lines.

2. The given equation is that of two tangents from (x', y'); hence, when (x', y') is at infinity, we obtain the equation of the tangents from infinity, in a given direction, which we find to be parallel, as evidently (from Geometry) they should be.

II. *Solution by* L. W. JONES, B.A.; Prof. EVANS, M.A.; *and others.*

Introducing the linear unit z, we may write the equation

$$(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)(ax'^2 + by'^2 + \&c.)$$
$$= [(ax' + hy' + gz')x + (hx' + by' + fz')y + (gx' + fy' + cz')z]^2.$$

When x', y' become infinite, and $lx' + my' = 0$, we have

$$\frac{x'}{m} = \frac{y'}{-l} = \frac{z'}{0}.$$

The equation thus becomes

$$(ax^2 + by^2 + c + 2fy + 2gx + 2hxy)(am^2 + bl^2 - 2hlm)$$
$$= [(am - hl)x + (hm - bl)y + gm - fl]^2.$$

2. The original equation is that of a conic having double contact with

$$ax^2 + by^2 + c + 2fy + 2gx + 2hyz = 0$$

at the extremities of the polar with respect to it of (x', y'). It will also be found that it has its centre at (x', y'). Hence the modified equation is that of the parabola which has double contact with the conic $ax^2 + by^2 + \dots + 2hxy = 0$ at the extremities of the diameter which bisects chords parallel to $lx + my = 0$.

III. *Solution by the* PROPOSER; Professor MOREL; *and others.*

Put $-\dfrac{lx}{m}$ for y; then we have

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy$$
$$= \frac{\left\{(ax + hy + g)x'(hx + by + f)\dfrac{lx'}{m} + (gx + fy + c)\right\}^2}{\left(ab + \dfrac{l^2}{m^2} - 2h\dfrac{l}{m}\right)x'^2 + 2\left(g - f\dfrac{l}{m}\right)x' + c},$$

which $= \dfrac{\infty}{\infty}$ when $x' = \infty$. Differentiating numerator and denominator each twice, we have $\dfrac{[m(ax + hy + g) - l(hx + by + f)]^2}{am^2 + bl^2 - 2hlm}$.

Multiplying up, and reducing, we have

$$Cl^2x^2 + Cm^2y^2 + Bm^2 + 2Hlm + Al^2$$
$$-2y\,(Glm - F'm^2) - 2x\,(Flm + Gl^2) + 2Clmxy = 0,$$

or $\quad Al^2 + Bm^2 + C\,(lx+my)^2 - 2Fm\,(lx+my) - 2Gl\,(lx+my) + 2Hlm = 0\,;$

or, if we write $-n$ for $lx+my$,

$$(A, B, C, F, G, H)\,(l, m, n)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Taking (1) to represent the equation to the pair of tangents from (x', y') to the conic $(a, b, c, f, g, h), (x, y, z)^2 = 0$, the equation (2) will represent the tangential equation of the same conic.

5571. (By Christine Ladd.)—Find the locus of a point with regard to which the reciprocal of a fixed triangle has a constant area, provided the radius of the auxiliary circle remains constant.

I. *Solution by the* Proposer; J. O'Regan; J. Hammond, M.A.; *and others.*

Taking the original triangle as triangle of reference, the distances from the point (P suppose) to the angular points of the reciprocal triangle are $\dfrac{k^2}{\alpha}, \dfrac{k^2}{\beta}, \dfrac{k^2}{\gamma}$; and the area of the reciprocal triangle is therefore proportional to $\dfrac{\sin A}{\beta\gamma} + \dfrac{\sin B}{\gamma\alpha} + \dfrac{\sin C}{\alpha\beta}$, or inversely proportional to $\alpha\beta\gamma$.

The locus of P is therefore the curve $\alpha\beta\gamma =$ constant, whose asymptotes are the sides of the original triangle, and whose three real inflexions range on the line at infinity. [Under a more extended form, this question was proposed by the Editor as Quest. 5161, and two geometrical solutions thereof may be seen on pp. 80, 81 of Vol. XXVII. of the *Reprint.*]

II. *Solution by* Vincenzo Jacobini.

Se il triangolo dato si prende come fondamentale e le perpendicolari sopra i suoi lati come coordinate del punto variabile; allora indicando con R il raggio del circolo ausiliario, con a, b, c i lati del triangolo fisso, A, B, C gli angoli opposti, a', b', c' i lati del reciproco sarà

$$a'^2 = R^4\left(\frac{1}{y^2} + \frac{1}{z^2} + \frac{2\cos A}{yz}\right),\quad b'^2 = R^4\left(\frac{1}{x^2} + \frac{1}{z^2} + \frac{2\cos B}{xz}\right),$$
$$c'^2 = R^4\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{2\cos C}{xy}\right),$$

quindi se S è l'area del triangolo reciproco, che vogliamo costante, tenendo

conto che

$$A+B+C=\pi, \text{ e che } 16S^2 = 2a'^2b'^2+2b'^2c'^2+2c'^2a'^2-a'^4-b'^4-c'^4,$$

si troverà $\quad xyz = (x \operatorname{sen} A + y \operatorname{sen} B + z \operatorname{sen} C)\dfrac{R^4}{2S} = \dfrac{4\Delta^2}{abc}\cdot\dfrac{R^4}{2S}$,

essendo Δ l'area del triangolo dato. Dunque il luogo richiesto è una curva del 3° ordine con tre punti d'inflessione a distanza infinita su i lati del triangolo.

Altra dimostrazione:—

Se ha
$$2S = \frac{R^2}{y}\cdot\frac{R^2}{z}\operatorname{sen} A + \frac{R^2}{x}\cdot\frac{R^2}{z}\operatorname{sen} B + \frac{R^2}{x}\cdot\frac{R^2}{y}\operatorname{sen} C$$
$$= R^4\left(\frac{\operatorname{sen} A}{yz}+\frac{\operatorname{sen} B}{xz}+\frac{\operatorname{sen} C}{xy}\right),$$

a dunque $\quad xyz = (x \operatorname{sen} A + y \operatorname{sen} B + z \operatorname{sen} C)\dfrac{R^4}{2S} = \dfrac{2\Delta^2 R^4}{abcS}$,

come sopra.

5339. (By Hugh McColl, B.A.) — In the quadratic equation $x\theta^2+z\theta+y=0$, the coefficient x is taken at random between 0 and 3, the coefficient y between -1 and 4, and the coefficient z between -3 and 3; show that the chance that the following events will simultaneously happen, namely, that x will be the greatest (algebraically) of the three coefficients, and that the roots of the equation will be real, is

$$\tfrac{1}{45}\left(\log 2 + \tfrac{43}{8}\right) = \cdot 196495\ldots.$$

Solution by the Proposer.

For the principles on which this solution is based, see the articles entitled "Symbolical Language" on pp. 20—23 and 100 of Vol. XXVIII. of the *Reprint*. The application of this method is not restricted to probability; the process will be found equally effective in ascertaining (without the assistance of curves or diagrams) the new limits which arise when we change the order of integration, or the variables, in a multiple integral. That the process may be clearly understood, I give this example fully worked out; but for the future the table of limits and two or three of the leading steps will be quite sufficient.

Def. 9.—The symbol x_0 asserts that zero is an *inferior* limit of x, and the symbol $x_{0'}$ asserts that zero is a *superior* limit of x. Hence $x_0 = px$, $x_{0'} = p'x$. Similarly we interpret y_0, $y_{0'}$, &c.

Def. 10.—The expression "factor A = B" means, not necessarily that the statement B is always an equivalent for A, but only that B may replace A in the particular compound statement of which A is a factor. For example, $p(xy) = x_0y_0 + x_{0'}y_{0'}$; but if we can see by inspection that either $x_{0'}$ or $y_{0'}$ is inconsistent with some co-factor of $p(xy)$, we may say "factor $p(xy) = x_0y_0$."

DEF. 11.—The symbol $A\iiint dx\,dy\,dz$ denotes the value of the integral when the variables are subject to the restrictions imposed by the statement A.

Table of Limits.		
$y_1 = -1$	$z_1 = -3$	$x_1 = 3$
$y_2 = 4$	$z_2 = 3$	$x_2 = \frac{3}{2}$
$y_3 = x$	$z_3 = x$	
$y_4 = \dfrac{z^2}{4x}$	$z_4 = 2x$	
	$z_5 = -2x$	

We now proceed to the solution.

Let A denote the given statement $x_{1'.0}\,y_{2'.1}\,z_{2'.1}$; and let Q denote the compound statement of the truth of which we are uncertain, namely, the statement $p'(y-x)\,p'(z-x)\,p'(4xy-z^2)$; that is to say, let $Q = y_{3'}z_{3'}\,p'(4xy-z^2)$.

$$\text{The required chance} = \frac{AQ\iiint dx\,dz\,dy}{A\iiint dx\,dz\,dy} = \frac{AQ\iiint dx\,dz\,dy}{90},$$

for the statement A is elementary, so that

$$A\iiint dx\,dz\,dy = \int_0^{x_1} dx \int_{z_1}^{z_2} dz \int_{y_1}^{y_2} dy = 90.$$

Now $$AQ = y_{3'.2'.1}\,z_{3'.2'.1}\,x_{1'.0}\,p'(4xy-z^2);$$

and $$\text{factor } p'(4xy-z^2) = p'\left(y-\frac{z^2}{4x}\right) = y_{4'}.$$

Hence $$AQ = y_{3'.2'.4'.1}\,z_{3'\,2'.1}\,x_{1'.0}.$$

But factor $y_{3'.2'} = y_{3'}$, and factor $z_{3'.2'} = z_{3'}$ by mere inspection of the table, and without employing Rule 3; hence $AQ = y_{3'.4'.1}\,z_{3'.1}\,x_{1'.0}$.

But factor $y_{3'.4'} = y_{3'}\alpha + y_{4'}\beta$ (by Rule 3), in which

$$\begin{aligned}\alpha &= p'(y_3-y_4) = p'\left(x-\frac{z^2}{4x}\right) = p'(4x^2-z^2) = p(z^2-4x^2)\\ &= p(z-2x)\,p(z+2x) + p'(z-2x)\,p'(z+2x)\\ &= p(z-2x) + p'(z+2x) = z_4 + z_{5'} = z_{5'},\end{aligned}$$

since z_4 is inconsistent with the co-factor $z_{3'}x_0$;

and $$\begin{aligned}\beta &= p'(y_4-y_3) = p'\left(\frac{z^2}{4x}-x\right) = p'(z^2-4x^2)\\ &= p'(z-2x)\,p(z+2x) + p(z-2x)\,p'(z+2x)\\ &= p'(z-2x)\,p(z+2x) = z_{4'.5}.\end{aligned}$$

Substituting, we get

$$\begin{aligned}AQ &= y_1\,z_{3'.1}\,x_{1'.0}\,(y_{3'}z_{5'} + y_{4'}z_{4'.5}) = y_{3'.1}\,z_{3'.5'.1}\,x_{1'.0} + y_{4'.1}\,z_{3'.4'.1.5}\,x_{1'.0}\\ &= y_{3'.1}\,z_{5'.1}\,x_{1'.0} + y_{4'.1}\,z_{3'.1.5}\,x_{1'.0}, \text{ by inspection.}\end{aligned}$$

In the second term, the factor $z_{1.5} = z_1\alpha + z_5\beta$ (by Rule 4), in which

$$\alpha = p(z_1-z_5) = p(-3+2x) = p(x-\tfrac{3}{2}) = x_2;$$

and $$\beta = p(z_5-z_1) = p(-2x+3) = p'(x-\tfrac{3}{2}) = x_{2'}.$$

Substituting, we get

$$\text{AQ} = y_{3'.1}\,z_{5'.1}\,x_{1'.0} + y_{4'.1}\,z_{3'}\,(z_1 x_2 + z_5 x_{2'})\,x_{1'.0}$$
$$= y_{3'.1}\,z_{5'.1}\,x_{1'.0} + y_{4'.1}\,z_{3'.1}\,x_{1'.0\,.2} + y_{4'.1}\,z_{3'.5}\,x_{2'.1'.0}$$
$$= y_{3'.1}\,z_{5'.1}\,x_{1'.0} + y_{4'.1}\,z_{3'.1}\,x_{1'.2} \;\; + y_{4'.1}\,z_{3'.5}\,x_{2'.0},$$

by mere inspection of the table.

We must now apply Rule 5 to each of these three terms thus :—

$$y_{3'.1} = y_{3'.1}\,a, \quad \text{where} \quad a = p\,(y_3 - y_1) = p\,(x+1) = 1,$$
$$z_{5'.1} = z_{5'.1}\,a, \quad \text{where} \quad a = p\,(z_5 - z_1) = p\,(-2x+3) = p'(2x-3)$$
$$= p'(x - \tfrac{3}{2}) = x_{2'}.$$

Hence the first term $= y_{3'.1}\,z_{5'.1}\,x_{2'.1'.0}$, which $= y_{3'.1}\,z_{5'.1}\,x_{2'.0}$ by inspection. This term is now *elementary*, and we will denote it by E_1.

Note.—A statement of the above form is said to be *elementary* when the application of Rule 5 to any of the variables will introduce either no fresh limits among the other variables, or only such as are implied in the limits already stated.

The second and third terms may be shown in like manner to be already elementary. We will denote them by E_2 and E_3 respectively; so that we have

$$\text{E}_1 = y_{3'.1}\,z_{5'.1}\,x_{2'.0}, \quad \text{E}_2 = y_{4'.1}\,z_{3'.1}\,x_{1'.2}, \quad \text{E}_3 = y_{4'.1}\,z_{3'.5}\,x_{2'.0};$$

and the required chance $= \frac{1}{90}\,(\text{E}_1 + \text{E}_2 + \text{E}_3)\iiint dx\,dz\,dy$

$$= \tfrac{43}{240} + \tfrac{1}{40}\log_e 2 = \cdot 196495 \,\ldots.$$

[This Solution, though, at the Proposer's request, subsequent in publication, was written before the second article on "Symbolical Language," which appeared on p. 100 of Vol. XXVIII. of the *Reprint*. A more regular order of the letters will be obtained by interchanging y and z both in the Question and Solution. The present order was adopted because the Proposer was at first under the erroneous impression that the shortest solution would be obtained by making the coefficient of θ vary first; and the first solution sent to the Editor was in accordance with this order of integration.]

5515. (By E. P. Culverwell, M.A.)—If r be one of the four normal distances of a point P from an ellipse, and p the parallel central perpendicular on a tangent line; prove that, if $\Sigma\left\{(pr)^{-1}\right\}$ vanishes, then P lies on the director circle of the ellipse; and state the corresponding theorem for an ellipsoid.

Solution by E. W. Symons; *the* Proposer; *and others.*

Let the coordinates of P be (α, β), and of Q, the foot of the normal, be (x', y'); then we have

$$a^2\left(\frac{x'-a}{x'}\right)=b^2\left(\frac{y-y'}{y'}\right)=\lambda;\qquad x'-a=\frac{x'}{a^2}\lambda;$$

$$r^2=(x'-a)^2+(y'-\beta)^2=\lambda^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)=\frac{\lambda^2}{p^2};\quad \lambda=\tfrac{1}{2}pr;$$

$$x^2(a^2+pr)=a^2a,\ \text{ therefore }\ \frac{x'}{a}=\frac{aa}{a^2+pr},\ \text{ and similarly for }\ \frac{y'}{b}.$$

Substituting in the ellipse, we have

$$a^2a^2(b^2+pr)^2+b^2\beta^2(a^2+pr)^2=(a^2+pr)(b^2+pr).$$

If the coefficient of pr in this equation vanish, the sum of the reciprocals vanishes. That coefficient is

$$[pr][2(a^2b^2a^2+b^2\beta^2a^2)-2(a^2b^4+b^2a^4)].$$

Hence sum of reciprocals vanishes if $a^2b^2[a^2+\beta^2-(a^2+b^2)]=0$, that is to say, if a, β lies on the director circle.

In the ellipsoid the sum of reciprocals evidently vanishes when (a,β) lies on an ellipsoid.

5467. (By CHRISTINE LADD.)—If three conics touch each other and have a common focus, prove that the common tangent of any two will cut the directrix of the third in three points which lie on one straight line.

I. *Solution by* S. JOHNSTON; C. BICKERDIKE; *and others.*

Reciprocating the theorem in Question 5356 with respect to any point, we have the theorem in question.

II. *Solution by* H. W. HARRIS, D. EDWARDES, *and others.*

First projecting and then reciprocating, we reduce the question to the following:—"Three conics touch each other at a common point C, and have two points A and B common to all; prove that the three polars of the line AB with respect to each of the conics lie on a right line."

Take ABC for triangle of reference, and let the three conics be

$$\frac{l_1}{\alpha}+\frac{m_1}{\beta}+\frac{n_1}{\gamma},\quad \frac{l_2}{\alpha}+\frac{m_2}{\beta}+\frac{n_2}{\gamma},\quad \frac{l_3}{\alpha}+\frac{m_3}{\beta}+\frac{n_3}{\gamma};$$

then, since they have a tangent at C (which we take as α,β) common, we have

$$\frac{l_1}{m_1}=\frac{l_2}{m_2}=\frac{l_3}{m_3}.$$

But the three polars of AB, or γ, have for their coordinates

$$(l_1, m_1, -n_1),\quad (l_2, m_2, -n_2),\quad (l_3, m_3, -n_3);$$

and the condition that these should lie on a right line is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0,$$

which is evidently satisfied if the relations above hold good.

5299. (By L. H. ROSENTHAL.)—Solve the simultaneous equations,

$$x^3-ax^2+(b-2y)\,x+ay-c=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$x^2y-axy-(y^2-by+d)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solution by the PROPOSER; J. RICHARDS, M.A.; *and others.*

To solve these equations, take the general biquadratic, viz.,

$$x^4-ax^3+bx^2-cx+d=0,$$

and assume it to be the product of two quadratic factors, viz.,

$$x^2-\lambda x+\mu \quad\text{and}\quad x^2-\nu x+\rho;$$

then, equating coefficients, we have the four equations,

$$\lambda+\nu=a,\;\; \lambda\nu+\mu+\rho=b,\;\; \mu\nu+\rho\lambda=c,\;\; \mu\rho=d \ldots\ldots (1,2,3,4),$$

between any three of which we can linearly eliminate either of the pairs λ, μ, or ν, ρ; the result being a relation between the other pair; and therefore, if we get two such relations from the four equations (1, 2, 3, 4), we shall have equations sufficient for the determination of these quantities. Now, eliminating ν and ρ between (1), (2), (3), and (1), (2), (4), we have respectively

$$\begin{vmatrix} 1 & 0 & \lambda-a \\ \lambda & 1 & \mu-b \\ \mu & \lambda & -c \end{vmatrix}=0, \qquad \begin{vmatrix} 1 & 0 & \lambda-a \\ \lambda & 1 & \mu-b \\ 0 & \mu & -d \end{vmatrix}=0;$$

or, expanding, $\quad \lambda^3-a\lambda^2+\lambda\,(b-2\mu)+a\mu-c=0,$

and $\quad \mu\lambda^2-a\mu\lambda-(\mu^2-b\mu+d)=0,$

respectively. Hence, from the foregoing, we see that the solutions of the two latter equations are $\lambda=\alpha+\beta$, $\mu=\alpha\beta$; where $\alpha, \beta, \gamma, \delta$ are the roots of $x^4-ax^3+bx^2-cx+d=0$. It is evident also from the above that there are six distinct (in one of the two equations) pairs of equations of the form proposed, whose solutions are the same.

Otherwise, to solve these equations directly, assume $x^2-ax+b=\Delta$, when the proposed equations become respectively,

$$x\Delta-y\,(2x-a)-c=0,\quad y\Delta-(y^2+d)=0 \;\ldots\ldots\ldots\ldots (5,6).$$

Solve for y from (5) in terms of x and Δ, substitute for it in (6), and eliminate x between result, and $x^2-ax+b-\Delta=0$; we thus get for Δ the equation $\Delta^3-b\Delta^2+(ac-4d)\,\Delta+4bd-c^2-a^2d=0$, the well-known equation for $\alpha\beta+\gamma\delta$, where $\alpha, \beta, \gamma, \delta$ are the roots of

$$x^4-ax^3+bx^2-cx+d=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

therefore $x^2-ax+b=\alpha\beta+\gamma\delta$, and hence, substituting for a and b in terms of the roots of (7), we find corresponding to the root $\alpha\beta+\gamma\delta$, $\lambda=\alpha+\beta$ and $\gamma+\delta$; similarly for the other two roots $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$. Again, from (6), substituting for Δ, $\alpha\beta+\gamma\delta$ we get for y the values $\alpha\beta$, $\gamma\delta$, and similarly for the other values of Δ.

Thus we see that the equation whose roots are $\alpha+\beta$ for a biquadratic may be written in the convenient and I believe new form,

$$(x^2-ax+b)^3-b\,(x^2-ax+b)^2+(ac-4d)\,(x^2-ax+b)+4bd-c^2-a^2d=0.$$

It is also evident that the equation in Δ is the same as the equation which would be obtained for $\mu+\rho$, from the four equations (1, 2, 3, 4), and thus answers an elimination question connected with those four equations.

Again, to obtain the equation in y whose roots are $\alpha\beta$, &c., we have, eliminating Δ between (5) and (6) and solving for x, $x = \dfrac{y(ay-c)}{y^2-d}$; substituting in (6) and reducing, we get for the required equation,

$$(y^2-d)^2(y^2-by+d)+y^2(ay-c)(cy-ad) = 0.$$

The same equations may be formed for an equation of the n^{th} degree by supposing it to be the product of a quadratic and an equation of the $(n-2)^{\text{th}}$ degree; and we can get two determinants of the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ degrees respectively in λ containing both λ and μ; between which eliminating either we get required equation in the other.

Similarly, by supposing an equation to be the product of two equations of the m^{th} and $(n-m)^{\text{th}}$ degrees, we can find the equations whose roots are $\Sigma_m\alpha$, $\Sigma_m\alpha\beta$, $\Sigma_m\alpha\beta\gamma$, &c. &c. This method, however, becomes very troublesome in equations beyond the sixth degree.

5441. (By D. Edwardes.)—Prove that

$$\frac{\sin 2\alpha\sin(\beta-\gamma)\cos(\theta-\alpha)+\sin 2\beta\sin(\gamma-\alpha)\cos(\theta-\beta)+\sin 2\gamma\sin(\alpha-\beta)\cos(\theta-\gamma)}{\sin 2\alpha\sin(\beta-\gamma)\sin(\theta-\alpha)+\sin 2\beta\sin(\gamma-\alpha)\sin(\theta-\beta)+\sin 2\gamma\sin(\alpha-\beta)\sin(\theta-\gamma)} \equiv \tan(\alpha+\beta+\gamma-\theta).$$

Solution by R. Tucker, M.A.; D. Edwardes; J. O. Jelly; *and others.*

Let $\theta \equiv \alpha+\beta+\gamma = \kappa$; then we have

$$4\,(\text{numerator}) = 4\left\{\sin 2\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\kappa)+\ldots+\ldots\right\}$$

$$= 2\sin 2\alpha\left\{\sin(2\beta-\kappa)-\sin(2\gamma-\kappa)\right\}+\ldots+\ldots$$

$$\begin{aligned} = {} & \cos(2\alpha-2\beta+\kappa)-\cos(2\alpha+2\beta-\kappa)-\cos(2\alpha-2\gamma+\kappa)+\cos(2\alpha+2\gamma-\kappa)\\ & +\cos(2\beta-2\gamma+\kappa)-\cos(2\beta+2\gamma-\kappa)-\cos(2\beta-2\alpha+\kappa)+\cos(2\beta+2\alpha-\kappa)\\ & +\cos(2\gamma-2\alpha+\kappa)-\cos(2\gamma+2\alpha-\kappa)-\cos(2\gamma-2\beta+\kappa)+\cos(2\gamma+2\beta-\kappa)\end{aligned}$$

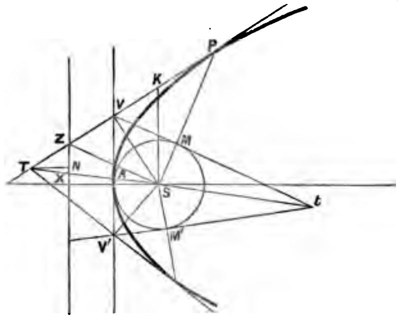
$$= -2\sin\kappa\left\{\sin(2\alpha-2\beta)+\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)\right\};$$

$$4\,(\text{denominator}) = 2\sin 2\alpha\left\{\cos(2\gamma-\kappa)-\cos(2\beta-\kappa)\right\}+\ldots+\ldots$$

$$= -2\cos\kappa\left\{\sin(2\alpha-2\beta)+\ldots+\ldots\right\};\ \text{therefore \&c.}$$

5475. (By C. Taylor, M.A.)—Prove the following construction for tangents to a conic:—Take a point T at a distance TN from the directrix, and divide ST in t so that $St : ST = AX : TN$, where A is the vertex, and X the foot of the directrix. About S draw a circle with radius SA, and from t draw tangents to the circle cutting the tangent at A in V, V′. Then TV, TV′ will touch the conic.

Solution by R. TUCKER, M.A.; S. TEBAY, B.A.; *and others.*



We have $\angle tSM = \angle tSM'$, $\angle VSM = \angle VSA = \angle V'ST$, $\angle VST = \angle V'SA = \angle V'SM'$, therefore TSt is a straight line; $\angle MVS = \angle AVS$ $= \angle SPT = \angle VSZ$, therefore VM is parallel to SZ; hence we have

$$\frac{TN}{AX} = \frac{TZ}{ZV} = \frac{ST}{St}.$$

[For clearness' sake, the figure has been drawn with tangents on opposite sides of the axis, T being the escribed centre of the triangle $VV't$.]

5564. (By Professor BENJAMIN PEIRCE, F.R.S.)—Find the probabilities at a game of a given number of points, which is played in such a way that there is only one person who is the actual player, and when the player is successful he counts a point, but when he is unsuccessful he loses all the points he has made and adds one to his opponent's score.

Solution by SEPTIMUS TEBAY, B.A.

The player may be successful a_1 times in succession, and then fail; after this he may be again successful a_2 times, and then fail a second time; and so on for i periods. Now, in order that the player may be finally successful, there must be $i-1$ failures, enabling the player to score a_i points. Therefore, if n be the number of points in the game, we must have

$$a_1 + a_2 + \ldots + a_i = n - i + 1.$$

The equation $\qquad a_1 + a_2 + \ldots + a_{i-1} = n - i + 1 - a_i,$

admits of $\dfrac{(n-1-a_i)!}{(i-2)!\,(n-i+1-a_i)!}$ solutions, each of which terminates with a_i successes. Hence the number of points scored by the player on this hypothesis is $a_i \dfrac{(n-1-a_i)!}{(i-2)!\,(n-i+1-a_i)!}$; and putting $n = 1, 2, \ldots n-i+1$, the general term is

$$\frac{1}{(i-2)!}\,.\,(n-i+2-x)\,.\,x\,(x+1)\,\ldots\,(x+i-3)$$

$$= \frac{1}{(i-2)!}\left[(n-i+1)\,.\,x\,(x+1)\,\ldots\,(x+i-3) - (x-1)\,x\,\ldots\,(x+i-3)\right]$$

and, integrating with respect to x, we have

$$\frac{n-i+1}{(i-1)!}\cdot(n-i+1)(n-i+2)\dots(n-1)-\frac{(n-i)(n-i+1)\dots(n-1)}{i\,(i-2)!}$$
$$=\frac{n!}{i!\,(n-i)!},$$

which is the number of points scored by the player in i periods. Putting $i=1, 2, \dots n$, the sum of the results is 2^n-1, which is the total number of points that can be scored by the player.

Now the whole number of ways in which the player can score is

$$\Sigma\left[\frac{(n-1-a_i)!}{(i-2)!\,(n-i+1-a_i)!}\right],\ \text{from } a_i=1 \text{ to } a_i=n-i+1,$$
$$=\frac{1}{(i-2)!}\,\Sigma\left[x(x+1)\dots(x+i-3)\right]=\frac{(n-1)!}{(i-1)!\,(n-i)!};$$

and in each of these there are $i-1$ failures, or $(i-1)\,\dfrac{(n-1)!}{(i-1)!\,(n-i)!}$ in all. Let $i=1, 2, \dots n$; then the number of failures

$$=1\,.\,(n-1)+2\,\frac{(n-1)(n-2)}{1\,.\,2}+3\,\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}+\dots+(n-1).$$

Now $\quad(1+x)^{n-1}=1+(n-1)\,x+\dfrac{(n-1)(n-2)}{1\,.\,2}\,x^2+\dots+x^{n-1}.$

Differentiating with respect to x, we have

$$(n-1)(1+x)^{n-2}=1\,.\,(n-1)+2\,\frac{(n-1)(n-2)}{1\,.\,2}\,x+\dots+(n-1)\,x^{n-2}.$$

Let $x=1$; therefore

$$(n-1)\,2^{n-2}=1\,.\,(n-1)+2\,\frac{(n-1)(n-2)}{1\,.\,2}+\dots+(n-1).$$

This result is obtained on the supposition that the player finally scores. But he can finally fail as often as he can finally succeed, namely, 2^n-1 times. Therefore the whole number of points that can be scored by the non-player is $\quad(n-1)\,2^{n-2}+2^n-1=(n+3)\,2^{n-2}-1.$

Hence their respective probabilities are

$$\frac{2^n-1}{(n+7)\,2^{n-2}-2},\quad\frac{(n+3)\,2^{n-2}-1}{(n+7)\,2^{n-2}-2}.$$

5621. (By D. Edwardes.)—If a circle be drawn through the centre of the inscribed circle and the centres of any two escribed circles of a triangle, prove that its radius is double that of the circumscribed circle of the triangle.

Solution by VINCENZO JACOBINI; W. S. F. LONG, B.A.; *and others.*

Sia ABC il triangolo, siano O, O_1, O_2, O_3 i centri del circolo inscritto e dei circoli ex-inscritti al me desimo; allora il circolo ABC sarà rispetto al triangolo $O_1O_2O_3$ il circolo dei nove punti, come il punto O sarà rispetto allo stesso il punto di concorso delle altezze, quindi il raggio del circolo ABC sarà metà del raggio del circolo $O_1O_2O_3$, ma quest' ultimo è eguale a quello dei circoli OO_1O_2, OO_2O_3, OO_1O_3, come si vede facilmente osservando che: triangoli $O_1O_2O_3$, OO_1O_2, OO_2O_3, OO_1O_3 hanno lo stesso circolo dei nove punti, dunque i circoli OO_1O_2, OO_1O_3, OO_2O_3 hanno i raggio doppio di quello del circolo ABC.

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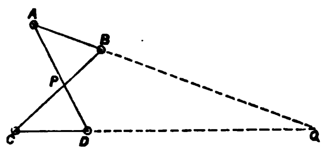
136. CONIC CONSTRUCTIONS. *By* E. J. LAWRENCE, M.A.

Take a frame ABCD, consisting of four links crossed as in the figure, in which AB = opposite link CD, and BC = opposite link AD. Suppose the latter pair of links the longer of the two. Then—



(1) If a short link CD be absolutely fixed, and the rest of the frame moved about it in the plane of the paper, the intersection P will describe an ellipse having C, D for foci, axis major = AD, and the tangent to the curve at P passes through Q the intersection of AB and CD produced.

(2) If a long link BC be absolutely fixed and the frame moved as before, Q, the intersection of the short links produced, will describe a hyperbola whose foci are B, C, its axis major = AB, and tangent to the curve at Q passes through P. The asymptotes will be parallel to the two directions in which AB and CD lie parallel to one another.

5573. (By Professor MONCK, M.A.)—The three edges and the diagonal of a rectangular parallelepiped are integer numbers; show how to obtain a series of parallelepipeds possessing the same quality.

Solution by the PROPOSER.

Let a, b, c be the edges of the parallelepiped, and d the diagonal. Another can be formed by taking for the new edges $a+b+d$, $a+c+d$, and $b+c+d$, and for the new diagonal $a+b+c+2d$; and the process can

be repeated *ad infinitum*. (Any of the edges can be taken as negative with a similar result.) For

$$(a+b+d)^2+(a+c+d)^2+(b+c+d)^2$$
$$= 2a^2+2b^2+2c^2+2ab+2bc+2ac+4(a+b+c)d+3d^2,$$

or (since $a^2+b^2+c^2=d^2$)

$$= a^2+b^2+c^2+2ab+2bc+2ac+2(a+b+c)(2d)+4d^2$$
$$= (a+b+c)^2+2(a+b+c)(2d)+(2d)^2 = (a+b+c+2d)^2.$$

The simplest parallelepiped of the kind is that whose edges are 1, 2, and 2, and diagonal 3; next comes that whose edges are 2, 3, and 6, and diagonal 7, which is derived from the former by the above rule, taking one of the 2's as negative.

5436. (By Dr. Booth, F.R.S.)—Express $\Sigma(\sec A)$ and $\Sigma(\operatorname{cosec} A)$ in terms of the radii of circles connected with the given triangle ABC.

Solution by the Editor.

From Booth's *Geometrical Methods* [Vol. II., pp. 294 (*e*), (*f*), 295 (*b*)],

$$\sin A \sin B+\sin B \sin C+\sin C \sin A = \frac{r^2+s^2+4Rr}{4R^2}\ \ldots\ldots\ldots\ldots(1),$$

$$\cos A\cos B+\cos B\cos C+\cos C\cos A = \frac{r^2+s^2-4R^2}{4R^2}\ \ldots\ldots\ldots(2);$$

$$\sin A\sin B\sin C = \frac{rs}{2R^2},\qquad \cos A\cos B\cos C = \frac{\rho}{2R}\ \ldots\ldots\ldots(3, 4).$$

Dividing (1) by (3), and (2) by (4), and bearing in mind that

$$r^2s^2 = rr_1r_2r_3,\quad s^2 = r_2r_3+r_3r_1+r_1r_2 = 4R^2+4Rr+r^2+2R\rho,$$

we have $$\Sigma\left(\frac{1}{\sin A}\right) = \frac{r^2+s^2+4Rr}{2rs} = \frac{2R^2+4Rr+r^2+R\rho}{(rr_1r_2r_3)^{\frac{1}{2}}},$$

and $$\Sigma\left(\frac{1}{\cos A}\right) = \frac{r^2+s^2-4R^2}{2R\rho} = \frac{2Rr+r^2+R\rho}{R\rho}.$$

5549. (By A. W. Panton, M.A.)—A plane cuts a spheroid of revolution, and makes an angle α with the axis; prove that the eccentricity of the section is equal to $e\cos\alpha$, where e is the eccentricity of a section through the axis.

Solution by R. E. RILEY, B.A.; R. F. DAVIS, B.A.; *and others.*

It will easily be seen that if a, b be the axes of the principal section, and a', b' those of an oblique section through the centre, then $b' = b$, and a' = that radius vector of the principal section making an angle α with the axis $= b(1-\epsilon^2\cos^2\alpha)^{\frac{1}{2}}$; therefore &c.

5492. (By M. HERMITE.)—Soit

$$F(x) = 1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} + \dots,$$

on demande de démontrer qu'on a

$$\frac{F(x)F(-x)}{n^2} = \frac{1}{n^2} + \frac{x^2}{n(n+1)^2(n+2)} + \frac{x^4}{n(n+1)(n+2)^2(n+3)(n+4)} + \dots$$

I. *Solution by* J. J. WALKER, M.A.; R. E. RILEY, B.A.; *and others.*

$$F'(x) = \frac{n}{x} + \left(1-\frac{n}{x}\right)F(x),\quad F'(-x) = -\frac{n}{x} + \left(1+\frac{n}{x}\right)F(-x) \;\dots(1).$$

Assume $F(x)F(-x) = u$, then $\frac{du}{dx} = F'(x)F(-x) - F(x)F'(-x)$,

or, by (1), $\frac{du}{dx} = \frac{n}{x}\{F(x)+F(-x)\} - \frac{2nu}{x}$, whence we have

$$x^{2n}\frac{du}{dx} + 2nx^{2n-1}u = 2nx^{2n-1}\left(1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)\dots(n+4)} + \dots\right).$$

Integrating, we obtain

$$x^{2n}u = c + x^{2n} + \frac{nx^{2n+2}}{(n+1)^2(n+2)} + \frac{nx^{2n+4}}{(n+1)(n+2)^2(n+3)(n+4)} + \dots$$

By putting $x=0$ the value of the arbitrary constant is found to be zero; and dividing the last equation by n^2x^{2n}, the result is as in the Question.

II. *Solution by* J. HAMMOND, M.A.; Professor BATTAGLINI; *and others.*

$$\frac{x^n F(x)}{\Gamma(n+1)} = \frac{x^n}{\Gamma(n+1)} + \frac{x^{n+1}}{\Gamma(n+2)} + \frac{x^{n+2}}{\Gamma(n+3)} + \dots = \phi, \text{ suppose};$$

$$\frac{x^n F(-x)}{\Gamma(n+1)} = \frac{x^n}{\Gamma(n+1)} - \frac{x^{n+1}}{\Gamma(n+2)} + \frac{x^{n+2}}{\Gamma(n+3)} - \dots = \psi, \text{ suppose};$$

$\frac{d\phi}{dx} - \phi = \frac{x^{n-1}}{\Gamma(n)}$, which, multiplied by e^{-x} and integrated, gives

$$\phi = \frac{e^x}{\Gamma(n)}\int_0^x e^{-x}x^{n-1}\,dx, \text{ and similarly } \psi = \frac{e^{-x}}{\Gamma(n)}\int_0^x e^x x^{n-1}\,dx;$$

whence $\dfrac{x^n F(x)}{n} = e^x \int_0^x e^{-x} x^{n-1}\,dx, \quad \dfrac{x^n F(-x)}{n} = e^{-x}\int_0^x e^x x^{n-1}\,dx;$

therefore $$\frac{x^{2n} F(x)\,F(-x)}{n^2} = \int_0^x\int_0^x e^{y-z}(yz)^{n-1}\,dy\,dz.$$

In this result put ux for y, and vx for z, and divide by x^{2n}, then

$$\frac{F(x)\,F(-x)}{n^2} = \int_0^1\int_0^1 e^{x(u-v)}(uv)^{n-1}\,du\,dv.$$

Differentiate this $2m$ times with respect to x, and put $x=0$ to get the $2m^{\text{th}}$ coefficient, then

$$\frac{F(x)\,F(-x)}{n^2} = \sum_{m=0}^{m=\infty}\left\{\frac{x^{2m}}{2m\,!}\int_0^1\int_0^1 (u-v)^{2m}(uv)^{n-1}\,du\,dv\right\}.$$

The general value of this definite integral for any value of m I have found to be $\dfrac{B(2m+1,\,n)}{m+n}$, but as the working is not elegant I have not given it.

Assuming this, then the general coefficient is

$$\frac{B(2m+1,\,n)}{(m+n)\,2m\,!} = \frac{\Gamma(n)}{(m+n)\,\Gamma(2m+n+1)}$$
$$= \frac{1}{n(n+1)\ldots(n+m)^2(n+m+1)\ldots(n+2m)}.$$

III. *Solution by* Robert Rawson; Professor Mannheim; *and others.*

Let $$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad (1),$$
therefore $$F(-x) = a_0 - a_1x + a_2x^2 - a_3x^3 + \ldots \qquad (2);$$
therefore $$F(x)\,F(-x) = A_0 + A_2x^2 + A_4x^4 + \ldots \qquad (3),$$
where
$$\left.\begin{aligned}
&A_0 = a_0^2,\ A_2 = 2a_2a_0 - a_1^2,\quad A_4 = 2a_4a_0 - 2a_3a_1 + a_2^2,\\
&A_6 = 2a_6a_0 - 2a_5a_1 + 2a_4a_2 - a_3^2,\\
&A_8 = 2a_8a_0 - 2a_7a_1 + 2a_6a_2 - 2a_5a_3 + a_4^2,\\
&\quad\vdots\\
&A_{2m} = 2a_{2m}\,.\,a_0 - 2a_{2m-1}\,.\,a_1 + 2a_{2m-2}\,.\,a_2 - 2a_{2m-3}\,.\,a_3 + \ldots \pm a_m^2
\end{aligned}\right\}\ (4),$$
the plus sign being used when (m) is even, and the minus sign when (m) is odd.

By the conditions of the question, we have

$$a_0 = 1,\quad a_1 = \frac{1}{n+1},\quad a_2 = \frac{a_1}{n+2},\quad a_3 = \frac{a_2}{n+3},\ \ldots\ldots\ a_{2m} = \frac{a_{2m-1}}{n+2m}.$$

Substituting these values in (4), we have

$$A_0 = 1,\ A_2 = a_2\left(2 - \frac{a_1^2}{a_2}\right) = a_2\left(2 - \frac{n+2}{n+1}\right) = \frac{na_2}{n+1},$$

$$A_4 = a_4\left(2 - \frac{2a_3a_1}{a_4} + \frac{a_2^2}{a_4}\right) = a_4\left(2 - \frac{2(n+4)}{n+1} + \frac{(n+3)(n+4)}{(n+1)(n+2)}\right) = -\frac{na_4}{n+2},$$

$$A_6 = a_6\left(2 - \frac{2a_5a_1}{a_6} + \frac{2a_4a_2}{a_6} - \frac{a_3^2}{a_6}\right)$$

$$= a_6\left(2 - \frac{2(n+6)}{n+1} + \frac{2(n+5)(n+6)}{(n+1)(n+2)} - \frac{(n+4)(n+5)(n+6)}{(n+1)(n+2)(n+3)}\right) = \frac{na_6}{n+3},$$

$$A_8 = a_8\left(2 - \frac{2a_7a_1}{a_8} + \frac{2a_6a_2}{a_8} - \frac{2a_5a_3}{a_8} + \frac{a_4^2}{a_8}\right)$$

$$= a_8\left(2 - \frac{2(n+8)}{n+1} + \frac{2(n+7)(n+8)}{(n+1)(n+2)} - \frac{2(n+6)(n+7)(n+8)}{(n+1)(n+2)(n+3)}\right.$$
$$\left. + \frac{(n+5)(n+6)(n+7)(n+8)}{(n+1)(n+2)(n+3)(n+4)}\right) = -\frac{na_8}{n+4}.$$

Similarly $A_{2m} = \pm\dfrac{na_{2m}}{n+m}$, according as (m) is odd or even.

From the above equations the property in the question is obvious.

By decomposing the coefficient of x^m, there results

$$\frac{F(x)-1}{x\epsilon^x} = \frac{1}{n+1} - \frac{x}{n+2} + \frac{x^2}{(n+3)\,2!} - \frac{x^3}{(n+4)\,3!} + \frac{x^4}{(n+5)\,4!} + \ldots$$

$$= B_0 + B_1\frac{n}{1} + B_2\frac{n^2}{2!} + B_3\cdot\frac{n^3}{3!} + B_4\cdot\frac{n^4}{4!} + \ldots,$$

where $\quad xB_0 = 1 - \epsilon^{-x},\ xB_1 = -\int B_0\,dx,\ xB_2 = -2\int B_1\,dx, \ldots\ldots,$

$$xB_m = -m\int B_{m-1}\,dx.$$

Hence $\quad \dfrac{F(x)-1}{\epsilon^x} = xB_0 - \int B_0\,dx\cdot\dfrac{n}{1} - \int B_1\,dx\cdot\dfrac{n^2}{1} - \int B_2\,dx\cdot\dfrac{n^3}{2!}$

$$-\int B_3\,dx\cdot\frac{n^4}{3!} - \ldots;$$

therefore

$$\left.\begin{aligned} &\frac{d}{dx}\left(\frac{F(x)-1}{\epsilon^x}\right) = \epsilon^{-x} - \frac{n}{x}\left(\frac{F(x)-1}{\epsilon^x}\right), \\ &F(x) = \frac{\int x^n\epsilon^{-x}\,dx}{x^n\epsilon^{-x}} + 1, \text{ and } F(-x) = 1 - \frac{\int x^n\epsilon^x\,dx}{x^n\epsilon^x} \end{aligned}\right\} \ldots\ldots (a).$$

The forms (a) do not appear to facilitate the method of obtaining the property in the question.

$u = F(x)$ satisfies the differential equation

$$\frac{d^2u}{dx^2} + \left(\frac{n}{x} - 1\right)\frac{du}{dx} + \frac{n}{x^2}(1-u) = 0;$$

and $u' = F(-x)$ satisfies the differential equation

$$\frac{d^2u'}{dx^2} + \left(\frac{n}{x} + 1\right)\frac{du'}{dx} + \frac{n}{x^2}(1-u') = 0.$$

5374. (By R. TUCKER, M.A.)—Prove that

$$\frac{1}{2^{2n-1}}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n-1}\,d\theta = \mathrm{B}(n+1,\,n),$$

where B is the symbol of the first Eulerian integral.

Solution by J. HAMMOND, M.A.; *the* PROPOSER; *and others.*

$$\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n-1}\,d\theta = \tfrac{1}{2}\int_0^{\pi}(\sin\theta)^{2n-1}\,d\theta;$$

therefore, putting on the right-hand side $\sin^2\frac{1}{2}\theta = x$, $dx = \sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\,d\theta$,

$$\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n-1}\,d\theta = 2^{2n-2}\int_0^1 x^{n-\frac{1}{2}}(1-x)^{n-\frac{1}{2}}\frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

$$= 2^{2n-2}\int_0^1 x^{n-1}(1-x)^{n-1}\,dx = 2^{2n-2}\,\mathrm{B}(n,n).$$

This agrees with the result in the question, for

$$\mathrm{B}(n+1,\,n) = \frac{\Gamma(n+1)\,\Gamma(n)}{\Gamma(2n+1)} = \frac{n\,\Gamma(n)\,\Gamma(n)}{2n\,\Gamma(2n)}.$$

5610. (By J. HAMMOND, M.A.)—Prove that

$$u \equiv \int_0^1 \frac{dx}{1-x^2}\log\left(\frac{2}{1+x^2}\right) = \frac{\pi^2}{16}.$$

I. *Solution by* Professor WOLSTENHOLME, M.A.

$$u = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\cos\theta}\log\left(\frac{2}{2-\cos^2\theta}\right) = -\int\frac{d\theta}{\cos\theta}\log\left(1-\tfrac{1}{2}\cos^2\theta\right)$$

$$= \int d\theta\left(\frac{1}{2}\cos\theta+\frac{1}{2}\cdot\frac{1}{2^2}\cos^3\theta+\frac{1}{3}\cdot\frac{1}{2^3}\cos^5\theta+\ldots\ldots\right)$$

$$= \frac{1}{2}+\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{2\,.\,4}{3\,.\,5}\cdot\frac{1}{2^3}+\ldots\ldots$$

$$= \left\{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right\}^2 = \frac{\pi^2}{16}.$$

II. *Solution by* R. TUCKER, M.A.

Let $x = \tan\frac{1}{2}\theta$, then $u = \frac{1}{2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\cos\theta}\log(1+\cos\theta)$.

Now if
$$v = \int_0^{\frac{1}{2}\pi}\frac{\log(1+\sin a\cos x)}{\cos x}\,dx,$$

then $$\frac{dv}{da} = \int_0^{\frac{1}{2}\pi} \frac{\cos a\, dx}{1+\sin a \cos x} = \tfrac{1}{2}\pi - a,$$

that is $$v = \tfrac{1}{2}\pi a - \tfrac{1}{2}a^2, \text{ therefore } u = \tfrac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) = \frac{\pi^2}{16}.$$

(See WILLIAMSON's *Integral Calculus*, Art. 113, 2nd ed.)

5599. (By the EDITOR.)—Calling the straight line that passes through the feet of the perpendiculars on the sides of a triangle from a point (P) on its circumscribed circle the SIMSON line of the point P, (from the name of the geometer who seems to have been the first to mention it), and putting l_1, l_2, l_3 for the segments of the SIMSON line that are included within the angles A, B, C of the triangle, respectively, and $a^2+b^2+c^2 = 2\sigma^2$,

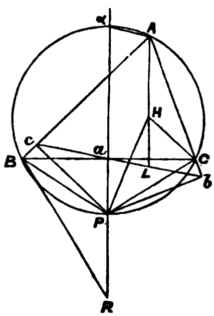
prove that $$(a^2-l_1^2)^{\frac{1}{2}} = (b^2-l_2^2)^{\frac{1}{2}} + (c^2-l_3^2)^{\frac{1}{2}} \text{ ....(1)},$$

$$(\sigma^2-a^2)\, l_1^2 + (\sigma^2-b^2)\, l_2^2 + (\sigma^2-c^2)\, l_3^2 = (2s)^2 \text{ ....(2)},$$

$$(\sigma^2-a^2)\, a^2 . \overline{\mathrm{AP}}^2 + (\sigma^2-b^2)\, b^2 . \overline{\mathrm{BP}}^2 + (\sigma^2-c^2)\, c^2 . \overline{\mathrm{CP}}^2 = a^2b^2c^2 \text{ ...... (3)}.$$

5600. (By CHRISTINE LADD.)—Required the envelop of the SIMSON line.

I. *Solution by* R. F. DAVIS, M.A.



1. If P be a point on the circle circumscribing the triangle ABC, and Pa, Pb, Pc be drawn perpendicular to the sides, then abc are in a straight line. For, joining ab, ac, circles may be made to circumscribe the quadrilaterals PacB, PaCb; therefore $\pi - \mathrm{P}ac = \mathrm{PBA} = \mathrm{PC}b = \mathrm{P}ab$.

This theorem is generally attributed to Dr. ROBERT SIMSON, the celebrated translator of Euclid's Elements.

The line abc may be termed the SIMSON (or pedal) line corresponding to P.

2. Let $bc = l_1$, $ca = l_2$, $ab = l_3$. Then PA $= l_1$ cosec A, &c. Consequently, if p be the extremity of the diameter through P,

$$p\mathrm{A}^2 = 4\mathrm{R}^2 - l_1^2 \operatorname{cosec}^2 \mathrm{A} = (a^2 - l_1^2) \operatorname{cosec}^2 \mathrm{A};$$

or $$p\mathrm{A} \sin \mathrm{A} = (a^2-l_1^2)^{\frac{1}{2}}.$$

But $p\mathrm{A} \sin \mathrm{A} \pm p\mathrm{B} \sin \mathrm{B} \pm pc \sin \mathrm{C} = 0$, hence $(a^2-l_1^2)^{\frac{1}{2}} + \ldots = 0$.

Expanding, we have $(a^2-l_1^2)^2 + \ldots - 2\,(b^2-l_2^2)\,(c^2-l_3^2) - \ldots = 0$,

or $$16s^2 - 4\,(\sigma^2-a^2)\, l_1^2 - \ldots + 16\sigma^2 = 0,$$

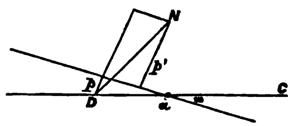
where s = area of triangle ABC, and σ = area of triangle whose sides are $l_1 l_2 l_3 = 0$. Therefore $(\sigma^2-a^2)\, l_1^2 + \ldots = (2s)^2$. The third relation is found by putting $l_1 = \mathrm{PA} \sin \mathrm{A}$, &c.

3. Let Pa, Pb, Pc be produced to meet the circumference again in $\alpha\beta\gamma$. Then $Pa\mathrm{A} = \mathrm{PBA} = ca\alpha$, and $\mathrm{A}\alpha$ is parallel to ac. Thus, $\mathrm{A}\alpha$, $\mathrm{B}\beta$, $\mathrm{C}\gamma$ are each parallel to the Simson line abc.

4. If $\mathrm{P}\alpha$, $\mathrm{P}'\alpha'$ be parallel chords each perpendicular to BC, and P, P′ extremities of a diameter, then also will α, α' be extremities of a diameter, and $\mathrm{A}\alpha$, $\mathrm{A}\alpha'$ at right angles. The Simson lines, therefore, corresponding to the extremities of any diameter are at right angles to each other.

5. Let H, R be the orthocentres of the triangles ABC, PBC respectively, and AH meet abc in L. Then $\mathrm{P}a\,.\,\mathrm{R}a = a\mathrm{B}\,.\,a\mathrm{C} = \mathrm{P}a\,.\,a\alpha$; hence $\mathrm{R}a = a\alpha = \mathrm{AL}$. But the distance of the orthocentre of any triangle from an angular point is twice the distance of the centre of the circumscribing circle from the opposite side; so that $\mathrm{PR} = \mathrm{AH}$. Therefore $\mathrm{P}a = \mathrm{HL}$; PH is bisected by abc. If a parabola then, whose focus is at P, be inscribed in the triangle ABC, abc is its tangent at the vertex, and consequently the orthocentre H lies on the directrix.

6. Finally, to find the envelop of abc. Let abc be inclined to BC at an angle ω. Then, if p be the perpendicular upon it from D, the middle point of BC,



$$p = \mathrm{D}a \sin\omega.$$

But
$$\mathrm{D}a = \mathrm{R}\sin(\mathrm{PCB}-\mathrm{PBC}) = \mathrm{R}\sin(2\omega+\mathrm{B}-\mathrm{C});$$

therefore
$$p = \mathrm{R}\sin(2\omega+\mathrm{B}-\mathrm{C})\sin\omega.$$

Now if p' be the perpendicular from N, the centre of the nine-points, circle upon abc, we have

$$p' = \tfrac{1}{2}\mathrm{R}\sin(\omega+\mathrm{NDC})-p = \tfrac{1}{2}\mathrm{R}\cos(\omega+\mathrm{B}-\mathrm{C})-p,$$

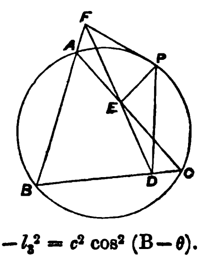
since $\mathrm{NDC} = \tfrac{1}{2}\pi+\mathrm{B}-\mathrm{C}$. Thus we obtain

$$p' = \tfrac{1}{2}\mathrm{R}\left\{\cos(\omega+\mathrm{B}-\mathrm{C})-2\sin(2\omega+\mathrm{B}-\mathrm{C})\sin\omega\right\} = \tfrac{1}{2}\mathrm{R}\cos(3\omega+\mathrm{B}-\mathrm{C}).$$

The envelop is therefore a three-cusped hypocycloid concentric with the nine-point circle.

II. *Solution by* W. J. C. Sharp, B.A.; Prof. Evans, M.A.; *and others*.

If O be the centre of the circle, and $\angle\,\mathrm{AOP} = 2\theta$, we have



$$\mathrm{AP} = 2\mathrm{R}\sin\theta,$$
$$\mathrm{FE} = 2\mathrm{R}\sin\theta\sin\mathrm{A} = a\sin\theta,$$
$$\mathrm{ED} = 2\mathrm{R}\sin(\mathrm{B}-\theta)\sin\mathrm{C} = c\sin(\mathrm{B}-\theta),$$
$$\mathrm{DF} = 2\mathrm{R}\sin(\mathrm{A}+\mathrm{B}-\theta)\sin\mathrm{B} = 2\mathrm{R}\sin(\mathrm{C}+\theta)\sin\mathrm{B} = b\sin(\mathrm{C}+\theta);$$

$$\therefore\quad a^2-l_1^2 = a^2\cos^2\theta,\quad b^2-l_2^2 = b^2\cos^2(\mathrm{C}+\theta),\quad c^2-l_3^2 = c^2\cos^2(\mathrm{B}-\theta).$$

Again
$$b\cos(\mathrm{C}+\theta)+c\cos(\mathrm{B}-\theta) = \cos\theta\,(b\cos\mathrm{C}+c\cos\mathrm{B})+\sin\theta\,(c\sin\mathrm{B}-b\sin\mathrm{C}) = a\cos\theta,$$

therefore
$$(b^2-l_2^2)^{\frac{1}{2}}+(c^2-l_3)^{\frac{1}{2}} = (a^2-l_1^2)^{\frac{1}{2}}.$$

Again, $\quad 2\left[(\sigma^2-a^2)\,l_1^2+(\sigma^2-b^2)\,l_2^2+(\sigma^2-c^2)\,l_3^2\right]$

$$= 2bc\cos A\times a^2\sin^2\theta+2ac\cos B\times b^2\sin^2(C+\theta)+2ab\cos C\times c^2\sin^2(B-\theta)$$

$$= 2abc\left\{a\sin^2\theta\cos A+b\sin^2(C+\theta)\cos B+c\sin^2(B-\theta)\cos C\right\}$$

$$= abcR\left\{\sin 2A\,(1-\cos 2\theta)+\sin 2B\,[1-\cos 2(C+\theta)] \right.$$
$$\left. +\sin 2C\,(1-\cos 2(B-\theta)]\right\}$$

$$= abcR\left\{\sin 2A+\sin 2B+\sin 2C-\cos 2\theta\,(\sin 2A+\sin 2B\cos 2C \right.$$
$$\left. +\cos 2B\sin 2C)+\sin 2\theta\,(\sin 2B\sin 2C-\sin 2B\sin 2C)\right\}$$

$$= abcR\,(\sin 2A+\sin 2B+\sin 2C)$$

$$= 4abcR\sin A\sin B\sin C$$

$$= \frac{a^2b^2c^2}{S}\times\frac{2S}{bc}\times\frac{2S}{ac}\times\frac{2S}{ab} = 2\,(2S)^2\,;$$

therefore $\quad (\sigma^2-a^2)\,l_1^2+(\sigma^2-b^2)\,l_2^2+(\sigma^2-c^2)\,l_3^2=(2S)^2.$

Again, $\quad (\sigma^2-a^2)\,a^2\,.\,AP^2+(\sigma^2-b^2)\,b^2\,.\,BP^2+(\sigma^2-c^2)\,c^2\,.\,CP^2$

$$= bc\cos A\,.\,a^2\,.\,4R^2\sin^2\theta+ca\cos B\,.\,b^2\,.\,4R^2\sin^2(C+\theta)$$
$$+ab\cos C\,.\,c^2\;4R^2\sin^2(B-\theta)$$

$$= 4abcR^2\left\{a\cos A\sin^2\theta+b\cos B\sin^2(C+\theta)+c\cos C\sin^2(B-\theta)\right\}$$

$$= 4\times\frac{abcR^3}{2}\left\{\sin 2A\,(1-\cos 2\;)+\sin 2B\,[1-\cos 2(C+\theta]\right.$$
$$\left.+\sin 2C\,[1-\cos 2(B-C)]\right\}$$

$$= 4\times\frac{abcR^3}{2}(\sin 2A+\sin 2B+\sin 2C)$$

$$= 8abcR^3\sin.A\sin B\sin C = 8abc\,.\,\frac{a^3b^3c^3}{64S^3}\times\frac{2S}{bc}\,.\,\frac{2S}{ac}\,.\,\frac{2S}{ab} = a^2b^2c^2.$$

III. *Solution by* CHRISTINE LADD, ELIZABETH BLACKWOOD, *and others.*

Let $l\alpha+m\beta+n\gamma=0$ be the equation to any transversal referred to the triangle $\alpha\beta\gamma$; then the condition that it should be the Simson line is that the perpendiculars to the sides of the triangle at their points of intersection with the line meet in a point. The equations to those perpendiculars are—

$$(n\cos B+m\cos C)\,\alpha+m\beta+n\gamma=0,$$
$$(l\cos C\;+n\cos A)\;\beta+\;n\gamma+\;l\alpha=0,$$
$$(m\cos A+l\cos B)\;\;\gamma+\;l\alpha+m\beta=0\,;$$

and the condition that these lines meet in a point is

$$\begin{vmatrix} n\cos B+m\cos C, & m, & n \\ l, & l\cos C+n\cos A, & n \\ l, & m, & m\cos A+l\cos B \end{vmatrix}=0,\text{ or}$$

$$2\lambda\mu\nu\,(1+\cos A\cos B\cos C)-\mu\nu\sin^2 A\,(\nu\cos B+\mu\cos C)$$
$$-\nu\lambda\sin^2 B\,(\lambda\cos C+\nu\cos A)-\lambda\mu\sin^2 C\,(\mu\cos A+\lambda\cos B)=0,$$

which is the tangential equation to the required envelop, and designates a three-cusped hypocycloid produced by rolling a circle of radius $\frac{1}{2}$R in a circle of radius $\frac{3}{2}$R concentric with the nine-point circle.

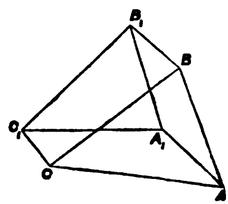
A cusp of the hypocycloid may be determined in the following manner:—If N′ is the foot of the perpendicular from N on BC, D the middle point of BC, M a point on BC such that $\angle N'NM=\frac{1}{3}\angle N'ND$, then NM produced through N passes through a cusp. For the tangents to the

curve through N are given by making $\cos(3\omega + B - C) = 0$, that is, $3\omega + B - C = 90°$ or $270°$ or $450°$, $\omega = 30° - \frac{1}{3}(B-C)$ or $90° - \frac{1}{3}(B-C)$ or $150° - \frac{1}{3}(B-C)$. Taking the second value, we have $N'NM = 90° - \omega = \frac{1}{3}(B-C)$, and it is well known that $N'ND = B - C$.

[Other properties of the SIMSON line, together with its envelop, have been investigated by Professors HIRST, TOWNSEND, and CLIFFORD, and published in former volumes of our *Reprints*. See Vol. III., pp. 58—59, pp. 81—83; Vol. IV., pp. 13—17; &c.]

5527. (By W. S. B. WOOLHOUSE, F.R.A.S.)—Project a given triangle orthogonally into an equilateral triangle.

Solution by R. RAWSON.



Let ABC be the given triangle and $A_1B_1C_1$ its orthogonal projection. Then AA_1, BB_1, CC_1 are parallel and perpendicular to the plane $A_1B_1C_1$. If the line AA_1 is fixed in magnitude and position, then BB_1, CC_1 are also fixed in magnitude and position. Let AA_1 make angles α, β with AC, AB respectively; these angles fix the position of AA_1.

Because AA_1 is perpendicular to the plane $A_1B_1C_1$ it is also perpendicular to the lines A_1B_1, A_1C_1.

Then $$A_1C_1 = AC \sin\alpha, \text{ and } A_1B_1 = AB \sin\beta \quad\text{..................... (1)},$$

$$BB_1 = AA_1 - AB\cos\beta, \text{ and } CC_1 = AA_1 - AC\cos\alpha,$$

therefore $$BB_1 - CC_1 = AC\cos\alpha - AB\cos\beta.$$

Hence, $$B_1C_1^2 + (AC\cos\alpha - AB\cos\beta)^2 = BC^2 \quad\text{..................... (2)}.$$

From (1), $AC\cos\alpha = (AC^2 - A_1C_1^2)^{\frac{1}{2}}$, and $AB\cos\beta = (AB^2 - A_1B_1^2)^{\frac{1}{2}}$.

Substitute these values in (2), then we have

$$(AC^2 - A_1C_1^2)^{\frac{1}{2}} = (AB^2 - A_1B_1^2)^{\frac{1}{2}} + (BC^2 - B_1C_1^2)^{\frac{1}{2}} \quad\text{............... (3)}.$$

This equation obtains for all angles α and β, which can be determined to satisfy only two independent conditions.

It is, therefore, inferred that the triangle ABC can be projected orthogonally into a triangle of any given species.

Put $A_1C_1 = A_1B_1 = B_1C_1 = x$, then (3) becomes

$$(AC^2 - x^2)^{\frac{1}{2}} = (AB^2 - x^2)^{\frac{1}{2}} + (BC^2 - x^2)^{\frac{1}{2}} \quad\text{................ (4)}.$$

From (4), by solving an ordinary quadratic, we obtain

$$3x^2 = AC^2 + AB^2 + BC^2$$
$$\pm 2(AC^4 + AB^4 + BC^4 - AC^2 . AB^2 - AC^2 . BC^2 - AB^2 . BC^2)^{\frac{1}{2}} \ldots (5).$$

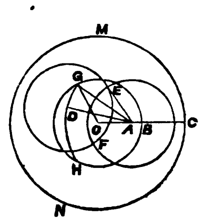
The angles α and β can be obtained by (1); and, if necessary, a geometrical construction could readily be obtained from (5).

[Mr. WOOLHOUSE remarks that he cannot regard this as a satisfactory solution, seeing that what is wanted is an elegant construction with ruler and compasses.]

5268. (By E. B. SEITZ.)—Two equal circles, each of radius r, are drawn on the surface of a circle of radius $2r$; show that the average area common to the two circles is $\left(1-\frac{16}{3\pi^2}\right)\pi r^2$.

Solution by the PROPOSER.

Let CMN be the circle of radius $2r$; O its centre; A, D the centres of the two equal circles; E, F their intersections. With O as a centre, and a radius equal to r, draw the circle BGH, and from A draw the arc GDH. Now A and D must be in the surface of the circle BGH. Put $OC = 2r$, $OB = AE = r$, $AD = AG = x$, $OA = y$, $\angle EAD = \theta$, $\angle OAG = \phi$, $\angle AOG = \psi$, $\angle AGO = \omega$, area common to the circles A and $D = \mu$. Then $x = 2r\cos\theta$,



$$\mu = 2r^2(\theta - \sin\theta\cos\theta),$$

$$\cos\phi = \frac{x^2+y^2-r^2}{2xy}, \quad \cos\psi = \frac{r^2+y^2-x^2}{2ry} \text{..(1, 2)},$$

$$\cos\omega = \frac{r^2+x^2-y^2}{2rx}, \quad y\sin\phi = r\sin\omega, \quad y\cos\psi = r - x\cos\omega \text{...(3, 4, 5)}.$$

Regarding x as a constant, and less than r, let A move over every point of the circle, centre O and radius $r-x$, and, for each position of A, D may move over the circumference of the circle of centre A and radius x. But when $x<r$ and $y>r-x$, or when $x>r$ and $y>x-r$, for each position of A, D may move over the arc GDH. Hence, since the whole number of ways the two circles can be drawn is $\pi^2 r^4$, we have, putting A for the required average

$$\pi^2 r^4 A = \int_0^r \int_0^{r-x} \mu \,.\, 2\pi x\,dx \,.\, 2\pi y\,dy + \int_0^r \int_{r-x}^r \mu \,.\, 2\phi x\,dx \,.\, 2\pi y\,dy$$

$$+ \int_r^{2r} \int_{x-r}^r \mu \,.\, 2\phi x\,dx \,.\, 2\pi y\,dy$$

$$= 2\pi^2 \int_0^r \mu (r-x)^2 x\,dx + 2\pi \int_0^r \int_{r-x}^r \mu x\,dx \,.\, 2\phi y\,dy + 2\pi \int_r^{2r} \int_{x-r}^r \mu x\,dx \,.\, 2\phi y\,dy.$$

But $\int 2\phi y\,dy = \phi y^2 - \int y^2 d\phi$. From (1) and (2),

$$\sin\phi\,d\phi = -\left(\frac{r^2+y^2-x^2}{2xy^2}\right)dy = -\frac{r\cos\psi\,dy}{xy} \text{..............(6)}.$$

From (4) and (6), $\quad x\sin\omega\,d\phi = -\cos\psi\,dy$ ....(7),
and, from (3), $\quad rx\sin\omega\,d\omega = y\,dy$ ....(8).
From (7) and (8), $\quad y\,d\phi = -r\cos\psi\,d\omega$ .... (9),
and from (5) and (9), $\quad y^2 d\phi = -r(r - x\cos\omega)\,d\omega$ .... (10),

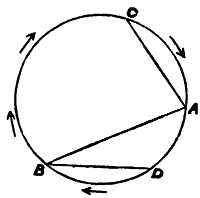
therefore $\quad \int 2\phi y\,dy = \phi y^2 + \omega r^2 - rx\sin\omega.$

Hence, observing that when $y = r-x$, $\phi = \pi$ and $\omega = 0$, when $y = r-x$, $\phi = 0$ and $\omega = 0$, and when $y = r$, $\phi = \omega = \theta$, we find

$$A = \frac{8}{\pi}\int_0^{2r} (\theta - \sin\theta\cos\theta)\,x\,dx = \frac{32r^2}{\pi}\int_0^{\frac{1}{2}\pi} (\theta - \sin\theta\cos\theta)^2 \sin\theta\cos\theta\,d\theta = \text{\&c.}$$

5602. (By Professor MONCK, M.A.)—Two chords of given length are drawn at random within a given circle; find the chance that they will intersect within the circle.

Solution by the PROPOSER.



Draw AB equal to the longer of the two chords, and from the ends A, B, draw AC, BD each equal to the shorter chord; taking care to let each of them be drawn in the direction given in the figure. Then, if we begin at A, and draw a chord equal to the shorter chord, in the direction indicated by the arrows, from every point on the circumference of the circle the chord so drawn will not intersect AB within the circle until we reach the point D. While passing along the arc DB the intersection will always take place inside the circle; but when we reach B it will again fall outside the circle until we arrive at C, where it will again fall inside the circle while we pass through the arc CA. The chance of intersection is therefore arc DB + arc CA : entire circumference; or, since arc DB = arc CA = arc subtended by the smaller chord, the chance of intersection is as twice this arc to the circumference.

[Mr. RILEY and others give the following solution of the problem:—"Fix one of the chords ($2a$); then the other ($2b$) envelops a circle of radius $(r^2-b^2)^{\frac{1}{2}}$ concentric with the given circle. Draw tangents to the inner circle from the ends of the fixed chord; then the proposed event happens whenever the moving tangent passes between two tangents drawn to the inner circle from the ends of the fixed chord, towards opposite parts." By this method they find, for the required probability,

$$\frac{1}{\pi r^4}\sin^{-1}\left\{2b\,(r^2-a^2)\,(r^2-b^2)^{\frac{1}{2}}+2a\,(r^2-b^2)\,(r^2-a^2)^{\frac{1}{2}}\right\},$$

a result which Prof. MONCK states that he has found to lead to the same as his own Solution.]

4902. (By C. W. MERRIFIELD, F.R.S.)—Can a sphere be touched by more than twelve other equal spheres?

5558. (By C. LEUDESDORF, M.A.)—Given a number of equal-sized spheres in the closest possible contact; find how many each would touch.

Solution by C. W. MERRIFIELD, F.R.S.

It is clear that a sphere *can* have twelve equal spheres in external contact with it. If the inner sphere meets the twelve edges of a cube, and the twelve points of contact be taken for the points of contact with the external spheres, these will also touch one another by threes and fours, and the planes of contact will form a right rhombic dodecahedron. But this is not the closest contact possible, for if we take a regular dodecahedron for the solid formed by the planes of contact, which will touch the inner sphere

at the points of the inscribed icosahedron, it is easy to show that we can have twelve spheres, each larger than the central sphere, in contact with it and with one another, six by six. The corresponding circles on the inner sphere will have a spherical radius of 31° 43′.

The question, therefore, really turns upon whether there is room for thirteen spheres,—it is easy to show that there is not room for fourteen,—and this is partly a question of arrangement. Observe that the question is the same as that of drawing circles on the sphere, each with a spherical radius of 30°, and touching, but not cutting, one another.

(1). Consider small circles with their centres one at each pole, and five circles arranged round each. These will touch the equator, and their centres will be on the parallels of latitude 30° N. and S. Each will be comprised between two meridians, whose difference of longitude is 70° 32′, and the question is: whether, if these are all crowded together, there is room for a thirteenth circle with its centre on the equator. It is easily shown, by the solution of a right-angled triangle, that the thirteenth circle, on this arrangement, will have a radius not exceeding 17° 40′, which is insufficient. Observe that this small circle is opposite to four circles in contact about a point opposite its centre.

(2). Consider one small circle about one pole, corresponding to three small circles about the other, and nine other circles arranged symmetrically in threes, as follows:—

1	circle about S. pole		....	(A),
3	circles centre lat. 30° S.,	lon. 0°, 120°, 240°	....	(B),
3	„ „ „ 8° 13′ S.,	„ 60°, 180°, 300°	....	(C),
3	„ „ „ 15° 48′ N.,	„ 0°, 120°, 240°	....	(D),
3	„ „ „ 52° 44′ N.,	„ 60°, 180°, 300°	....	(E).

In this arrangement, the latitudes are calculated on the assumption that (A) and (B), and (B) and (C) are in contact, and so (D) and (E). It is easy then to prove, by solving a right-angled triangle, that (C) and (D) will clear; but it will also be obvious that (B) and (D) will not clear.

(3). It remains to be seen whether, assuming the system (A), (B), and (C) to remain fixed, we can solve the problem by causing (D) and (E) to revolve together round the N. pole so that (C) and (D) shall touch, and then see if there is clearance between (B) and (D). For this purpose it is not necessary to have recourse to spherical trigonometry,—although an oblique spherical triangle will answer the enquiry exactly,—as we may substitute "plain sailing" for spherical, to use a phrase borrowed from navigation. That is to say, we may take the minutes as miles, and set off the latitudes square from the equator. In this way it is easily shown that the possible revolution will not give us the clearance required, and therefore, *à fortiori*, we shall not have it in the sphere. Therefore, in this case also, there will not be room for thirteen circles.

No more compact arrangement can be devised. Hence, only twelve equal spheres can be in external contact with an equal sphere; but in such a system there will be room to spare.

5553. (By R. A. Roberts, B.A.)—Prove that the points of contact of parallel tangents to a Cartesian oval lie on a conic which passes through four fixed points.

Solution by J. W. Sharpe, B.A.; J. Hammond, M.A.; *and others.*

The equation to such a quartic can always be expressed in the form $C^2 + LN^3 = 0$; where $C = 0$ is a circle, $L = 0$ is the bitangent, and $N = 0$ is the line at infinity.

Now if the polar of the point $(x'y'z')$ with respect to the circle be $P = 0$, and L', N' be the results of substituting x', y', z' in L and N, we find that the first polar of $(x'y'z')$ with respect to the quartic is

$$2CP + 3N'N^2L + L'N^3 = 0.$$

Now let $(x'y'z')$ be a point at infinity, then the tangents drawn from it to the curve are parallel, and $N' = 0$. Hence we get $2CP + L'N^3 = 0$ as the curve which cuts the Cartesian in the points of contact of the tangents from $(x'y'z')$. Combining this with the equation to the Cartesian, we find that another such curve is $N^3(2PL - L'C) = 0$; rejecting the factor N^3 as irrelevant to the present problem, we get the conic $2PL - L'C = 0$. This contains the two undetermined ratios $x' : y' : z'$ in the first degree; but these are equivalent to only one, because they satisfy the equation to the line at infinity; therefore the conic contains one indeterminate of the first degree, and therefore passes through four fixed points.

Obviously we may thus generalize the theorem:—The tangents to a bicuspidal quartic, from any point on the line joining the cusps, touch the curve in points which lie on a conic, which conic passes through four fixed points.

5607. (By J. Royds, A.C.P.)—Find x from the equation

$$(a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}} = b.$$

Solution by J. O'Regan; D. J. McAdam; *and others.*

By obvious operations, we obtain, successively,

$$2a + 3b\,(a^2 - x^2)^{\frac{1}{3}} = b^3, \quad 3b\,(a^2 - x^2)^{\frac{1}{3}} = b^3 - 2a;$$

$$27b^3(a^2 - x^2) = (b^3 - 2a)^3, \text{ therefore } x = \left\{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3\right\}^{\frac{1}{2}}.$$

5542. (By Professor Minchin, M.A.)—A solid triangular prism is placed, with its axis horizontal, on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the prism.

Solution by G. S. Carr.

Let r = distance of the centroid of the prism from the edge upon which it begins to turn, and θ = the inclination of r to the vertical.

We shall assume that no appreciable angular momentum is communicated to the prism by tilting the plane.

Let γ be the indefinitely small angle which r must make with the vertical, after passing it, before the force of cohesion between the base of the prism and the plane, and the force of stationary friction between the edge of the prism and the plane, are overcome.

The edge of the prism is of the nature of a cylindrical surface, and it is obvious that, however small the forces alluded to may be, they are greater than the moving force due to gravity just before the commencement of the motion.

The prism will begin to roll under the force of gravity opposed by a small couple of rolling friction which will be sensibly constant in the beginning of the motion. Let this couple be denoted by L. The equation of angular momentum will be initially

$$m(r^2+k^2)\frac{d^2\theta}{dt^2} = mgr\theta - \mathrm{L}, \text{ or putting } \frac{gr}{r^2+k^2} = \mu \text{ and } \frac{\mathrm{L}}{m(r^2+k^2)} = c,$$

$$\frac{d^2}{dt^2}\left(\theta-\frac{c}{\mu}\right) = \mu\left(\theta-\frac{c}{\mu}\right), \text{ therefore } \theta-\frac{c}{\mu} = \mathrm{A}e^{t\mu^{\frac{1}{2}}}+\mathrm{B}e^{-t\mu^{\frac{1}{2}}}.$$

When $t=0$, we have $\theta=\gamma$, and $\frac{d\theta}{dt}=0$, therefore

$$\mathrm{A}=\mathrm{B}=\tfrac{1}{2}\left(\gamma-\frac{c}{\mu}\right), \text{ and } \theta = \frac{c}{\mu}+\tfrac{1}{2}\left(\gamma-\frac{c}{\mu}\right)\left(e^{t\mu^{\frac{1}{2}}}+e^{-t\mu^{\frac{1}{2}}}\right),$$

which determines the motion.

[Professor Minchin remarks that his question has not been answered, inasmuch as Mr. Carr has solved the question on the supposition that the equilibrium of the prism will be broken by a rolling motion, whereas he wants to know whether its equilibrium will be broken by slipping or rolling by the gradual tilting up of the plane.]

5594. (By Professor Townsend, F.R.S.)—Prove the following pairs of reciprocal properties of a system of two conics :—

(*a*) When two conics are such that two of their four common points subtend harmonically the angle determined by the tangents at either of the remaining two, they subtend harmonically that determined by those at the other also.

(*b*) When two conics are such that two of their four common tangents divide harmonically the segment determined by the points of contact of either of the remaining two, they divide harmonically that determined by those of the other also.

(*c*) The associated conic, envelope of the system of lines divided harmonically by the two original conics, breaks up, in the former case, into the point-pair determined by the eight tangents to them at their four common points.

(*d*) The associated conic, locus of the system of points subtended harmonically by the two original conics, breaks up, in the latter case, into the line-pair determined by their eight points of contact with their four common tangents.

I. *Solution by* J. J. Walker, M.A.; W. J. C. Sharp, M.A.; *and others.*

These properties may be proved by projection, or analytically, thus:— Let A, B, C, D be the four common points, and

$$fyz+gzx+hxy=0, \quad f'yz+g'zx+h'xy=0$$

the equations referred to the triangle ABC; then (a) the condition that DB, AB should be harmonic conjugates with respect to the tangents at B is readily found to be

$$2gg'(fh'-f'h)-(fg'+f'g)(gh'-g'h)=0,$$

or $$(fg'-f'g)(gh'+g'h)=0, \quad \text{or} \quad gh'+g'h=0,$$

rejecting the condition of contact, $fg'-f'g=0$. By symmetry $gh'+g'h=0$ would also be the condition that DC, AC should be harmonic conjugates with respect to the tangents at C.

Further, a point not specially stated as among the properties [though virtually involved in (c) and (d)], $gh'+g'h=0$ being also the condition that BA, CA should be harmonic conjugates with respect to the tangents at A, to (a) may be added, "and these remaining two subtend harmonically the angle between the tangents at either of the first two."

(c) The tangential equation of the envelope is obviously

$$(f\alpha-g\beta-h\gamma)(f'\alpha-g'\beta-h'\gamma)=0.$$

Now the lines $g'x+f'y=0, \quad h'x+f'z=0, \quad hy+gz=0,$

and $$f(gh'-g'h)^2x+g(hf'-h'f)^2y+h(fg'-f'g)^2z=0$$

satisfy the relation $f'\alpha-g'\beta-h'\gamma=0$, when $gh'+gh'=0$. But the first pair of these lines are the tangents to the first conic at B, C; the second pair are the tangents to the other conic at A, D. Similarly, the other four tangents at ABCD meet in the point $f\alpha-g\beta-h\gamma=0$.

II. *Solution by the* Proposer.

The preceding all follow immediately from the following well-known pairs of reciprocal properties:—

(1) Every angle of a triangle inscribed to a conic is cut harmonically by the tangent at its vertex, and by its connector with the intersection of the tangents at the remaining two vertices. (1′) Every side of a triangle circumscribed to a conic is cut harmonically at its point of contact and at its intersection with the chord of contact of the remaining two sides. (2) The associated third conic, envelope of lines divided harmonically by a system of two conics, touches the eight tangents to the two at their four common points. (2′) The associated third conic, locus of points subtended harmonically by a system of two conics, passes through the eight points of contact with the two of their four common tangents. (3) When, of eight different tangents to the same conic, any three are concurrent, the conic touching the eight breaks up into a pair of points, of which that of the concurrence of the three is one. (3′) When, of eight different points on the same conic, any three are collinear, the conic containing the eight breaks up into a pair of lines, of which that of the collinearity of the three is one. From these well-known principles the four properties in the question are obvious and immediate consequences.

The two original conics being supposed to be both circles, it appears from (d) above, as stated by Mr. Panton in Quest. 5609, of which Quest. 5479 is obviously a particular case, that the associated conic, locus of points

subtending both harmonically, breaks up into a line-pair, when the two external intersect the two internal common tangents at right-angles. Since, then, every two of them, one external and the other internal, evidently divide harmonically each of the remaining two; and again, from (*c*) and (*d*) above, as shown on other principles in my *Modern Geometry*, Vol. I., Art. 208, that both associated conics break up, the envelope into a point-pair and the locus into a line-pair, when they intersect themselves at right-angles;—since then, evidently, their two imaginary common points subtend harmonically the angles at the two real determined by their lines of passage through them, and their two imaginary common tangents divide harmonically the segments of the two real intercepted between their points of contact with them.

5484. (By A. Martin, M.A.)—Find three positive integral numbers, the product of any two of which, diminished by the sum of the same two, shall be a square. [2, 8, 6 are *one* set of such numbers.]

I. *Solution by* L. W. Jones, B.A.; C. Vincenzo; *and others.*

Let x, y, z be the numbers; then

$$xy-x-y=\square, \quad \text{therefore} \quad (x-1)(y-1)=\square+1 \ldots\ldots\ldots\ldots (1).$$

Assume then $\quad x-1=p_1^2+q_1^2, \quad y-1=p_2^2+q_2^2;$

then $\quad (x-1)(y-1)=(p_1p_2+q_1q_2)^2+(p_1q_2-p_2q_1)^2.$

Equation (1) is therefore satisfied by the assumed values if $p_1q_2-p_2q_1=\pm 1$, that is, if $\dfrac{p_1}{q_1}, \dfrac{p_2}{q_1}$ are successive convergents to any continued fraction.

Similarly we may put $z-1=p_3^2+q_3^2$, if

$$p_3q_1-p_1q_3=\pm 1 \quad \text{and} \quad p_3q_2-p_2q_3=\pm 1.$$

Each of these two equations is satisfied if

$$p_3=p_1+p_2, \quad q_3=q_1+q_2.$$

And since $\dfrac{ab+1}{b}, \dfrac{abc+a+c}{bc+1}$ are successive convergents to a continued fraction, we may put (where a, b, c are any integers)

$$x=1+b^2+(ab+1)^2,$$
$$y=1+(bc+1)^2+(abc+a+c)^2,$$
$$z=1+(bc+b+1)^2+(abc+ab+a+c+1)^2.$$

Particular sets are (2, 3, 6), (3, 6, 14), (2, 6, 11), (6, 14, 35), (2, 18, 27).

II. *Solution by* C. Leudesdorf, M.A.; Prof. Evans, M.A.; *and others.*

We are to find three numbers x, y, z, such that

$$yz-y-z, \quad zx-z-x, \quad xy-x-y \quad \text{are all squares.}$$

Now $\quad x=a^2+b^2+1, \quad y=c^2+d^2+1, \quad z=e^2+f^2+1$

will be three such numbers, if only

$$ad-bc = \pm 1, \quad cf-de = \pm 1, \quad af-be = \pm 1 \;\ldots\ldots\ldots\ldots\ldots (\alpha);$$

for then
$$xy-x-y = (x-1)(y-1)-1$$
$$= (a^2+b^2)(c^2+d^2)-(ad-bc)^2 = (ac+bd)^2,$$

and similarly for the others.

If now $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ be three consecutive convergents to any continued fraction, the first two of the conditions (α) are satisfied; and if, in addition, the quotient corresponding to $\frac{e}{f}$ be 1, the third condition is also satisfied, since
$$af-be = a(b+d)-b(c+a) = ad-bc = \pm 1.$$

We have, then, the following Rule:—

Write down any continued fraction whatever which has 1 for one of its quotients, and form the convergent $\frac{e}{f}$ corresponding to that quotient, and also the two preceding convergents $\frac{c}{d}, \frac{a}{b}$: then will $a^2+b^2+1, c^2+d^2+1, e^2+f^2+1$ be three integers as required.

Taking, for example, the fraction $1+\frac{1}{1+}\frac{1}{1+}$, &c., whose convergents are $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}$, &c., we find the sets of numbers 2, 3, 6; 3, 6, 14; 6, 14, 35; 14, 35, 90; 35, 90, 234, &c.

5499. (By Professor Townsend, F.R.S.)—A solid circular cylinder of uniform density and infinite length, being supposed to attract, according to the law of the inverse sixth power of the distance, a material particle projected, with the velocity from infinity under its action, from any point external to its mass, in any direction perpendicular to its axis; show that the particle will describe freely, under its action, a circular arc orthogonal to the surface of the cylinder.

I. *Solution by* R. E. Riley, B.A.; E. W. Symons; *and others.*

Let a = radius of cylinder; and from P, the initial position of the particle, draw a perpendicular PO on the axis, and measure OZ $= z$ along the axis; let $\angle$ OPZ $= \theta$; then the attraction of an elementary section at Z on P is $\frac{\pi \cdot \mu \cdot a^2\, dz}{(\mathrm{PZ})^6}$; hence, resolving the attractions along OP, we get the whole attraction on the particle $= 2\mu \frac{\pi a^2}{r^5}\int_0^{\frac{1}{2}\pi} \cos^5\theta\, d\theta$, where r = OP; hence the

attraction of the cylinder on $P \propto r^{-5}$, and the particle, if projected from P perpendicular to PO, with the velocity from infinity, will move as if under a central force ($\propto r^{-5}$) at O; that is, it will describe an arc of the circle drawn on OP as diameter.

II. *Solution by the* PROPOSER.

The attraction of the cylinder for the law of the inverse sixth power of the distance being, as is well known, the same as if sixteen-fifteenths of the mass of its unit of length were concentrated by uniform longitudinal compression in its transverse section by the plane of motion of the particle, and attracted according to the law of the inverse fifth power of the distance instead; and a material particle moving freely in the plane of a uniform circular plate, with the velocity from infinity under its attraction for the latter law of the distance, describing, by SYLVESTER's theorem, a circular arc orthogonal to the circumference of the plate, therefore, &c.

5498. (By Professor MINCHIN, M.A.)—If E is the complete elliptic function of the second kind, with modulus k, and if k' is the complementary modulus, prove that, if n assume all values from 1 to ∞,

$$E = \tfrac{1}{2}(\pi k'^2)\left\{1 + \Sigma(2n+1)\left(\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}.k^n\right)^2\right\}.$$

I. *Solution by* R. RAWSON, CHRISTINE LADD, *and others.*

By reference to CAYLEY's *Elliptic Functions*, p. 46, we have

$$a_2 = \frac{1}{2^2},\quad a_4 = \frac{1.3}{2^2.4^2},\quad a_6 = \frac{1.3^2.5}{2^2.4^2.6^2},\quad a_8 = \frac{1.3^2.5^2.7}{2^2.4^2.6^2.8^2},\quad \ldots\ldots$$

$$\ldots\ldots\; a_{2n} = \frac{1.3^2.5^2\ldots 2n-1}{2^2.4^2.6^2\ldots(2n)^2};$$

$$\begin{aligned}
\therefore\; E &= \tfrac{1}{2}\pi\left\{1 - a_2k^2 - a_4k^4 - a_6k^6 - a_8k^8 - \ldots - a_{2n}k^{2n} - \&c.\right\}\\
&= \tfrac{1}{2}(\pi k'^2)\left\{1 - a_2k^2 - a_4k^4 - a_6k^6 - a_8k^8 - \&c.\right\} \div (1-k^2)\\
&= \tfrac{1}{2}(\pi k'^2)\left\{1 + (1-a_2)k^2 + (1-a_2-a_4)k^4 + (1-a_2-a_4-a_6)k^6 + \&c.\right\}\\
&= \tfrac{1}{2}(\pi k'^2)\left\{1 + \frac{1.3}{2^2}k^2 + \frac{1.3^2.5}{2^2.4^2}k^4 + \frac{1.3^2.5^2.7}{2^2.4^2.6^2}k^6 + \ldots\ldots\right.\\
&\qquad\left.\ldots\ldots\frac{1.3^2.5^2\ldots(2n-1)^2(2n+1)}{2^2.4^2.6^2\ldots(2n)^2}.k^{2n} + \ldots\right\}\\
&= \tfrac{1}{2}(\pi k'^2)\left\{1 + \Sigma(2n+1)\left(\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}k^n\right)^2\right\}.
\end{aligned}$$

This series does not converge so rapidly as the ordinary series, as given by Professor CAYLEY, above referred to.

The following series, obtained in a similar way to the above, possesses a greater degree of convergency than either:

$$E = \frac{\pi}{2k'^2}\left\{1 + \Sigma\,\frac{8n-3}{(2n-3)(2n-1)}\,a_{2n}k^{2n}\right\}.$$

II. *Solution by* J. HAMMOND, M.A.; J. O. JELLY, M.A.; *and others.*

The general term of the expansion in powers of k^2 of

$$\frac{\pi}{2}(1-k^2)\left\{1 + \frac{1^2.3}{2^2}k^2 + \frac{1^2.3^2.5}{2^2.4^2}k^4 + \dots\right\}$$

is

$$\tfrac{1}{2}\pi\left\{\text{coefft. of } k^{2n} - \text{coefft. of } k^{n2-2}\right\}k^{2n}$$

$$= \frac{\pi}{2}\left\{\frac{1^2.3^2.5^2\dots(2n-1)^2.(2n+1)}{2^2.4^2.6^2\dots(2n)^2} - \frac{1^2.3^2.5^2\dots(2n-3)^2.(2n-1)}{2^2.4^2.6^2\dots(2n-2)^2}\right\}k^{2n}$$

$$= k^{2n}\,\frac{\pi}{2}\left\{\frac{1^2.3^2.5^2\dots(2n-3)^2(2n-1)}{2^2.4^2.6^2\dots(2n-2)^2}\right\}\left(\frac{(2n-1)(2n+1)}{2n^2}-1\right)$$

$$= -\frac{\pi}{2}\,\frac{1^2.3^2.5^2\dots(2n-3)^2(2n-1)}{2^2.4^2.6^2\dots(2n)^2}k^{2n}.$$

The expansion is therefore

$$\frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2.3}{2^2.4^2}k^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}k^6 - \text{\&c.}\right\},$$

which is a well-known expression for E.

These expansions only give good results when k is small, since the series are divergent when $k = 1$ and only converge very slowly when k is near 1.

5563. (By Professor SYLVESTER, F.R.S.)—If at two points in a cubic curve, lying in a straight line with a point of inflexion, tangents be drawn to meet the curve again, prove that their intersections with it will also be in a straight line with the same point of inflexion.

I. *Solution by* W. J. C. SHARP, M.A.; Prof. CASEY, M.A.; *and others.*

It is shown in SALMON'S *Higher Plane Curves* (Art. 149), that the equation to any cubic may be thrown into the form $ABC = kD^2E$, where A, B, and C are tangents, D the chord of contact, and E its satellite.

Now, if D pass through an inflexion, and A be the inflexional tangent, $E = 0$ must also be satisfied when $A = 0$ and $D = 0$ are so, or E passes

through the point of inflexion. But E is the line joining the two other points in which B and C, the tangents at the points where D cuts the curve, meet the curve.

II. *Solution by* J. L. McKenzie, B.A.

This theorem, like that of Question 4869, may be proved by the principle of "residuation." (See Salmon's *Higher Plane Curves*, 2nd Ed., p. 133.) Take four fixed points on a given cubic, and draw through them any curve of the second order; the line joining the two remaining points in which the two curves intersect will pass through a fixed point on the cubic—the residual of the four fixed points. In the present question take for the four points, the points P and Q, at which the tangents are drawn, each twice. One system of the second order through the four points consists of the tangents PP and QQ, which meet the cubic again in the points X and Y; then line XY must pass through the point-residual. Another system of the second order through the four points is the line PQ taken twice; and this line meets the cubic again in I—the point of inflexion; hence the line II must pass through the point-residual. But II is the inflexional tangent, and the third point in which it meets the cubic coincides with I, which is therefore the point-residual. Therefore XY passes through I. [Another Solution is given on p. 52 of this volume.]

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138. Note on Question 5458. *By the* Editor. (See pp. 55—56 of this Volume.)

On p. 125 of his *Aperçu Historique des methodes en Géométrie* (Ed. 1875), M. Chasles states that "La Hire a donné le lieu des angles égaux, aigus ou obtus, circonscrits à une conique, lequel est une courbe du quatrième degré ... ;" that La Hire has also "traité la même question pour la cycloïde, et est parvenu à ce résultat curieux, savoir, que tous les angles égaux, droits, aigus, ou obtus, circonscrits à cette courbe, ont leurs sommets sur une seconde cycloïde raccourcie ou allongée ;" and that he himself has found "que les épicycloïdes du cercle jouissent de la même propriété ; c'est à dire que, Si à une epicycloïde, engendrée par un point d'une circonference de cercle qui roule sur un autre cercle fixé, on circonscrit des angles tous egaux entre eux, leurs sommets seront situés sur une épicycloïde allongée ou raccourcie."

5540. (By Professor Lloyd Tanner, M.A.)—If M, N are two numbers of n digits each, and the numbers formed by prefixing M to N and N to M are as a to b; find M, N, and indicate the conditions required to ensure (1) at least one solution, and (2) only one solution.

I. *Solution by* S. TEBAY, B.A.; Prof. EVANS, M.A.; *and others.*

The required statement is $\dfrac{N+10^n M}{M+10^n N} = \dfrac{a}{b}$; and, K being a conditional factor, we can take $KM = 10^n a - b$, $KN = 10^n b - a$. If n, a, b are given, these expressions may be prime to one another, and the solution impossible. In general K will be a factor of $10^{2n}-1$; and the number of solutions will depend upon the number of factors answering the conditions of the problem. If M, N be the least possible numbers, and M or $N > \frac{1}{3} \times 10^n$, there is but one solution; if less, there may be two solutions.

Let $n=1$, $a=19$, $b=25$; then $10 \times 19 - 25 = 5 \times 33$, $10 \times 25 - 19 = 7 \times 33$; here the only value of K is 33, and therefore $M=5$, $N=7$. Let $n=2$, $a=248$, $b=257$; therefore

$$10^2 \times 248 - 257 = 909 \times 27, \quad 10^2 \times 257 - 248 = 909 \times 28.$$

Take $K=909$, then $M=27$, $N=28$. Take $K=303$, then $M=81$, $N=84$.

II. *Solution by the* PROPOSER.

Let r be the radix, then $Mr^n + N : a = Nr^n + M : b$,

or $$(bM - aN)\, r^n = aM - bN, \text{ but } bM - aN = c \;\;\ldots\ldots\ldots(1, 2);$$

then, since M, N are each less than r^n, we have, by (1),

$$a > c > -b. \;\;\ldots\ldots\ldots(3).$$

Solving (1), (2), we have $M = \dfrac{ar^n - b}{a^2 - b^2}c$; $N = \dfrac{br^n - a}{a^2 - b^2}c$;

whence $$Mr^n + N = \frac{ac}{a^2-b^2}(r^{2n}-1); \quad Nr^n + M = \frac{cb}{a^2-b^2}(r^{2n}-1) \;\;\ldots\ldots\ldots(4).$$

If we assume $a > b$, and remember that M, N, r, a, b are all positive, it is clear that c is positive also, or (3) may be written $a > c > 0$ ....(5). Let us also suppose $a : b$ to be in its lowest terms, so that both a, b will be prime to $a^2 - b^2$. Then, since the expressions are integral, $a^2 - b^2$ must be a divisor of $c\,(r^{2n}-1)$; and c will be found in the following manner. Suppose α to be the denominator of $r^{2n}-1 : a^2 - b^2$ when reduced to its lowest terms, then c must be a multiple of α; and it may be any multiple of α which is less than a. Should α be greater than a, there can of course be no solution.

It is not difficult to determine (when a, b, r only are given) for what values of n, if any, the problem admits of solution. Consider $a^2 - b^2$ as composed of two factors; one, α, a product of divisors of r; the other, β, prime to r.

If $\alpha > a$ there is no solution for any value of n. For α is prime to $r^{2n}-1$, since this latter quantity is prime to r; hence α cannot divide $c\,(r^{2n}-1)$, since, by (5), $c < a, < \alpha$; hence $c\,(r^{2n}-1)$ is not divisible by $\alpha\beta$ or $a^2 - b^2$, and the expressions on the left of (4) cannot be integral.

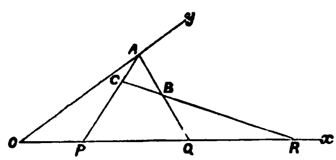
If, on the other hand, $\alpha < a$, we can always find values of n which make the problem soluble. For we may take c, in (4), to be any multiple of α less than a, and of such multiples there is at least one, viz. α itself. Also, β being prime to r, we can always find values for n such that $r^{2n}-1$ may be a multiple of β. [See a paper by Mr. GLAISHER, in the *Messenger of Mathematics*, Vol. V., p. 4.]

5580. (By S. Roberts, M.A.)—**If the sides of a variable triangle pass through three fixed points in a straight line, while one vertex moves on another straight line, and a second vertex describes a given curve, prove that the locus of the third vertex is a homographic transformation of the given curve.**

I. *Solution by* Professor Townsend, F.R.S.

Denoting by A, B, C the three vertices of the variable triangle, by P, Q, R the three collinear points through which its opposite sides by hypothesis pass, by L the right line on which its vertex A by hypothesis moves, and by E and F the two figures traced out by its two remaining vertices B and C; then, taking any two positions ABC and A′B′C′ of the variable triangle, since, by hypothesis, their three pairs of corresponding sides, BC and B′C′, CA and C′A′, AB and A′B′, intersect at three collinear points P, Q, R, therefore, by the fundamental property of triangles in homology in a plane, their three pairs of corresponding vertices, A and A′, B and B′, C and C′, connect by three concurrent lines L, M, N; hence, the two figures E and F, being such that all pairs of their corresponding points B and C connect through a fixed point P, and that all pairs of their corresponding lines M and N intersect on a fixed line L, are in homology in their plane to the centre P and axis L; and therefore, &c.

II. *Solution by the* Proposer; E. B. Elliott, M.A.; *and others.*



Take the line PQR through the fixed points for the axis of x, and the line directrix Oy for the axis of y on which the vertex A moves; the vertex B, whose coordinates are x, y, moves on the curve $F(x, y, 1) = 0$. Let X, Y be the coordinates of the third vertex C, and write k for OA, and a, b, c for OP, OQ, OR.

Then $$aY + (X-a)k = 0, \quad by + (x-b)k = 0;$$
or $$aY(x-b) - by(X-a) = 0.$$
Hence $aYx - b(X-a)y - abY = 0$, and $Yx - (X-c)y - cY = 0$;
giving $$x[(a-b)X + a(b-c)] = b(a-c)X,$$
$$y[(a-b)X + a(b-c)] = a(b-c)Y.$$
The required locus is $F[b(a-c)X,\ a(b-c)Y,\ (a-b)X + a(b-c)] = 0.$

5625. (By Professor Cayley, F.R.S.)—The equation
$$\{q^2(x+y+z)^2 - yz - zx - xy\}^2 = 4(2q+1)xyz(x+y+z)$$
represents a trinodal quartic curve having the lines $x=0$, $y=0$, $z=0$,

$x+y+z=0$ for its four bitangents; it is required to transform to the coordinates X, Y, Z, where $X=0$, $Y=0$, $Z=0$ represent the sides of the triangle formed by the three nodes.

Solution by J. J. WALKER, M.A.

Assume $x=\lambda X+Y+Z$, $y=X+\lambda Y+Z$, $z=X+Y+\lambda Z$, giving

$$x+y+z=(\lambda+2)(X+Y+Z),$$
$$yz+zx+xy=(2\lambda+1)(X^2+Y^2+Z^2)+(\lambda^2+2\lambda+3)(YZ+ZX+XY),$$
$$xyz=\lambda(X^3+Y^3+Z^3)+(\lambda^2+\lambda+1)(X^2Y+\dots)+(\lambda^3+3\lambda+2)XYZ;$$

then, by substitution, the given equation becomes

$$\left[\left\{(\lambda+2)^2q^2-(2\lambda+1)\right\}(X^2+Y^2+Z^2)+\left\{2(\lambda+2)^2q^2-(\lambda^2+2\lambda+3)\right\}(YZ+ZX+XY)\right]^2$$
$$=4(\lambda+2)(2q+1)\left\{\lambda(X^3+Y^3+Z^3)+(\lambda^2+\lambda+1)(X^2Y+\dots)+(\lambda^3+3\lambda+2)XYZ\right\}(X+Y+Z);$$

which, for the value $\lambda+2=-\dfrac{1}{q}$ or $\lambda=-\dfrac{2q+1}{q}$, becomes

$$\left\{\frac{2(2q+1)}{q}(X^2+Y^2+Z^2)-\left(\frac{q+1}{q}\right)^2(XZ+ZX+XY)\right\}^2$$
$$=\frac{4(2q+1)}{q}\left\{\frac{2q+1}{q}(X^3+Y^3+Z^3)-\frac{3q^2+3q+1}{q^2}(X^2Y+\dots)+\frac{12q^3+15q^2+6q+1}{q^3}XYZ\right\}(X+Y+Z).$$

It is evident that, on multiplying out, the coefficients of $X^4\dots$ are identically equal on both sides; viz., to $\dfrac{4(2q+1)^2}{q^2}$. Those of $X^3Y\dots$ are

$$-\frac{4(2q+1)(q+1)^2}{q^3}\text{ and }\frac{4(2q+1)}{q}\left(\frac{2q+1}{q}-\frac{3q^2+3q+1}{q^2}\right)$$

respectively, which are also equal. Hence the equation reduces to the form $\mu(Y^2Z^2+Z^2X^2+X^2Y^2)+\nu(X^2YZ+Y^2ZX+Z^2XY)=0$, and X, Y, Z are the lines joining the nodes. The values of μ and ν are $\left(\dfrac{3q+1}{q}\right)^4$ and $-2\dfrac{(3q+1)^3(q+1)}{q^4}$ respectively.

Hence the equation, transformed as required, is

$$(3q+1)(Y^2Z^2+Z^2X^2+X^2Y^2)-2(q+1)(X^2YZ+Y^2ZX+Z^2XY)=0.$$

5548. (By ELIZABETH BLACKWOOD.)—P, Q, R are three random points within a circle, and their respective distances from the centre are p, q, r; show that the chance that the roots of the equation $px^2-qx+r=0$ are real is $\frac{5}{144}+\frac{1}{12}\log_e 2$.

Solution by E. B. Seitz; A. Martin; *and others.*

The roots of the equation are real, if $r < \frac{q^2}{4p}$.

When $p < \frac{1}{4}a$ and $q < (4ap)^{\frac{1}{2}}$, R may be taken anywhere within the circle, radius $\frac{q^2}{4p}$, and concentric with the given circle, radius a; and when $p < \frac{1}{4}a$ and $q > (4ap)^{\frac{1}{2}}$, R may be taken anywhere within the given circle. When $p > \frac{1}{4}a$, from $q = 0$ to $q = a$, R may be taken anywhere within the circle, radius $\frac{q^2}{4p}$. Hence the chance that the roots are real

$$= \frac{1}{\pi^3 a^6}\int_0^{\frac{1}{4}a}\left\{\int_0^{(4ap)^{\frac{1}{2}}} \pi\left(\frac{q^2}{4p}\right)^2 . 2\pi q dq + \int_{(4ap)^{\frac{1}{2}}}^{a} \pi a^2 . 2\pi q\, dq\right\} 2\pi p\, dp$$

$$+ \frac{1}{\pi^3 a^6}\int_{\frac{1}{4}a}^{a}\int_0^{a} \pi\left(\frac{q^2}{4p}\right)^2 . 2\pi p dp . 2\pi q dq$$

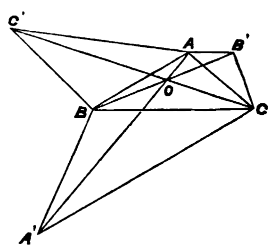
$$= \frac{2}{3a^3}\int_0^{\frac{1}{4}a}(3ap - 8p^2)\, dp + \tfrac{1}{24}\int_{\frac{1}{4}a}^{a} p^{-1}\, dp = \tfrac{5}{144} + \tfrac{1}{12}\log_e 2.$$

5588. (By T. Mitcheson, B.A., L.C.P.)—If on each side of a triangle ABC triangles similar thereto are drawn, so that the angles adjacent to A are each equal to C, those adjacent to B each equal to A, and those adjacent to C each equal to B: prove that three circles drawn round the outer triangles, and the three lines that join their vertical angles with the opposite angles of ABC, all intersect at a point O, such that the angles OBC, OCA, OAB are equal to each other; and that cot OBC = cot A + cot B + cot C.

5654. (By T. Mitcheson, B.A., L.C.P.)—If within a triangle a point O be taken such that $\angle$ ACO = BAO = CBO = θ, prove that

$$\cot\theta = \cot A + \cot B + \cot C.$$

Solution by Professor Morel; Professor Cochez; *and others.*



1. Les cercles circonscrits aux triangles ABC′ et ACB′, se coupent au point O, et l'on a

AOB + AC′B = AOC + AB′C = 180°.

On en déduit facilement d'après les données du problème,

BOC + BA′C = 180°.

Donc le cercle circonscrit au triangle BA′C passe par le point O.

2. L'angle AOB′ est égal à l'angle ACB′ et par suite égal à l'angle B; de même l'angle B′OC est égal à l'angle C, et l'angle COA′ est égal à l'angle A.

Donc la ligne AOA′ est une ligne droite, puisque la somme des trois angles est égale à 180°. On verrait de même que BOB′ est une ligne droite, et que COC′ est une ligne droite.

3. Les angles B′AC, ACB étant égaux, les droites AB′ et BC sont parallèles, donc l'angle AB′O est égal à l'angle OBC ; mais le quadrilatère inscrit AB′CO nous montre que l'angle AB′O est aussi égal à l'angle ACO. Donc ACO = OBC. On verrait de même que CBO = BAO.

4. Les triangles OAB, OAC donnent

$$\mathrm{OA} = \frac{c \sin(\mathrm{B}-\theta)}{\sin \mathrm{B}} = \frac{b \sin \theta}{\sin \mathrm{A}}, \quad \text{donc} \quad \cot \theta = \frac{b \sin \mathrm{B} + c \sin \mathrm{A} \cos \mathrm{B}}{c \sin \mathrm{A} \sin \mathrm{B}}.$$

Remplaçons b par la valeur égale $a \cos \mathrm{C} + c \cos \mathrm{A}$, puis, dans l'equation ainsi obtenue, au numérateur et au dénominateur, les cotés a et c par les quantités proportionnelles sin A et sin C, il vient, après réduction,

$$\cot \theta = \cot \mathrm{A} + \cot \mathrm{B} + \cot \mathrm{C}.$$

5613. (By R. A. Roberts, M.A.)—Prove that the locus of the centroid of a triangle inscribed in a conic and circumscribed to a parabola is a straight line.

Solution by R. F. Davis, B.A.; J. O'Regan; *and others.*

Let a triangle be inscribed in a circle and circumscribed to a parabola. The centre of the circumscribing circle being fixed and the orthocentre lying on the directrix, the centroid of the triangle which divides the straight line joining these two points in the ratio 2 : 1 will consequently describe a straight line. Projecting orthogonally, the required property is obtained.

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137. On the Sign of any Term of a Determinant. *By* G. R. Dick, M.A.

Let the determinant be written thus:—

$$\begin{vmatrix} a_1, & b_1, & \dots & l_1 \\ a_2, & b_2, & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ a_n, & b_n, & \dots & l_n \end{vmatrix} = \Delta,$$

and suppose we require the sign of the term $a_\alpha \, b_\beta \dots l_\lambda$.

Let the determinant be expanded in terms of the constituents of the first column and the corresponding minors, so that $\Delta = \Sigma a_\alpha \,.\, \mathrm{A}_\alpha$, where A_α is the minor formed by omitting the first column and the α^{th} row. In the first place, the sign of a_α is $(-1)^{\alpha-1}$; to find now the sign of b_β in the minor A_α, we remark that, if the suffix $\beta > \alpha$, its sign will be changed, but

if $\beta < \alpha$, will be unaffected; hence its sign will always be the same as that of $(-1)^{\beta-1}(\alpha-\beta)$. Proceeding in this way, we see that the sign of c_γ, in its containing minor, will be that of $(-1)^{\gamma-1}(\alpha-\gamma)(\beta-\gamma)$, and so on, till finally we get for the sign of $a_\alpha b_\beta \dots l_\lambda$,

$$\begin{aligned}\text{Sign of } &(-1)^{\alpha-1}.(-1)^{\beta-1}(\alpha-\beta).(-1)^{\gamma-1}(\alpha-\gamma)(\beta-\gamma)\dots\dots \\ &\qquad\dots\dots(-1)^{\lambda-1}(\alpha-\lambda)(\beta-\lambda)\dots(\kappa-\lambda) \\ &= (-1)^{\alpha+\beta+\dots\lambda-n}(\alpha-\beta)(\alpha-\gamma)\dots(\alpha-\lambda) \\ &\qquad\qquad(\beta-\gamma)\dots(\beta-\lambda) \\ &\qquad\qquad\qquad(\kappa-\lambda).\end{aligned}$$

But $$\alpha+\beta+\dots\lambda = 1+2+\dots n = \tfrac{1}{2}n(n+1);$$

hence we get the following result:—The sign of any term $a_\alpha, b_\beta, \dots l_\lambda$ of the given determinant is the same as the sign of the algebraical product,

$$(-1)^{\frac{1}{2}n(n-1)}(\alpha-\beta)(\alpha-\gamma)\dots(\alpha-\lambda)(\beta-\gamma)\dots(\beta-\lambda)\dots(\kappa-\lambda).$$

It is perhaps worth noting that the coefficient (± 1) of the term $a_\alpha . b_\beta \dots l_\lambda$ is the determinant formed by putting $a_\alpha, b_\beta, \dots l_\lambda$ each $= 1$, and all the remaining constituents each $= 0$.

5611. (By Prof. Wolstenholme, M.A.)—Having given that

$$\frac{e(y^2z^2+1)+(y^2+z^2)}{yz} = \frac{e(z^2x^2+1)+(z^2+x^2)}{zx} = \frac{e(x^2y^2+1)+(x^2+y^2)}{xy} = k,$$

prove that $k = e^2-1$, and that $yz+zx+xy = (yz)^{-1}+(zx)^{-1}+(xy)^{-1}$.

Solution by D. Edwardes; J. McDowell, M.A.; *and others*

From the two first equations we have $e = \dfrac{z^2-xy}{xyz^2-1}$; therefore

$$e = \frac{z^2-xy}{xyz^2-1} = \frac{x^2-yz}{yzx^2-1} = \frac{z^2-x^2+y(z-x)}{xyz(z-x)}, \quad e = \frac{x+y+z}{xyz} \dots \text{(A, B)},$$

$$\frac{z^2-xy}{xyz^2-1} = \frac{x^2-yz}{yzx^2-1} \text{ gives } xy+yz+zx = (xy)^{-1}+(zx)^{-1}+(yz)^{-1}.$$

Again, taking the two first equations, we have, by (B) and (A),

$$\begin{aligned}k &= \frac{ez^2(y^2-x^2)+y^2-x^2}{z(y-x)} = \frac{(ez^2+1)(y+x)}{z} = (ez^2+1)(exy-1) \\ &= e^2xyz^2+e(xy-z^2)-1 = e^2xyz^2+e^2(1-xyz^2)-1 = e^2-1.\end{aligned}$$

[Professor Wolstenholme remarks that the question possesses an interest for him as being the algebraical representation of the system

$$\begin{aligned}e\cos(\beta+\gamma)+\cos(\beta-\gamma) &= e\cos(\gamma+\alpha)+\cos(\gamma-\alpha) \\ &= e\cos(\alpha+\beta)+\cos(\alpha-\beta),\end{aligned}$$

the first system of poristic equations he happened to come across, which he deduced from the geometrical theorem that if an ellipse be inscribed in a triangle with a focus at the centre of the circumscribed circle, its

major axis is equal to the radius (R) of the circumscribed circle. The expression of this in polar coordinates (with focus for pole) gives the system. It is very easy to make any number of such poristic systems; for example, the three equations

$$\frac{(a-y)(a-z)}{\{1+(y+z)^2\}^{\frac{1}{2}}} = \frac{(a-z)(a-x)}{\{1+(z+x)^2\}^{\frac{1}{2}}} = \frac{(a-x)(a-y)}{\{1+(x+y)^2\}^{\frac{1}{2}}} = \lambda(1+4a^2)$$

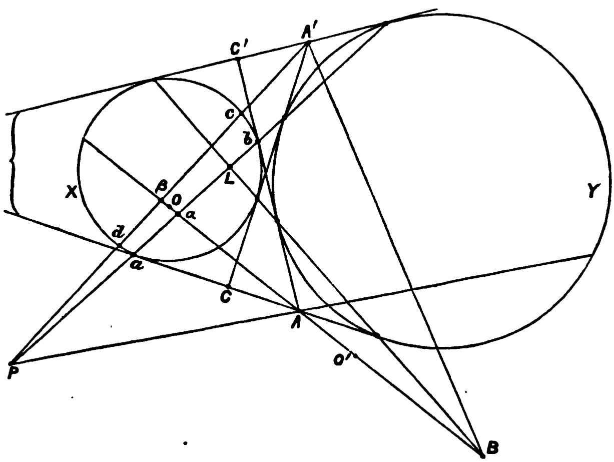
are poristic, and can only be satisfied if $\lambda^2 = 1$, being then equivalent to

$$yz+zx+xy-2a(x+y+z) = 1+a^2, \quad xyz+(1+3a^2)(x+y+z)+2a^3 = 0,$$

deducible from a triangle inscribed in a parabola, and circumscribed to a circle whose centre lies on the parabola.]

5609. (By A. W. Panton, M.A.)—The four common tangents to two circles being supposed such that the two opposite pairs, external and internal, are at right angles to each other; show that their eight points of contact with the circles lie on two straight lines, every point on each of which subtends the circles in an harmonic system of tangents.

Solution by W. S. M'Cay, M.A.



For any two circles, the polars of A and A′ intersect at a limiting point; but on account of the right angles at C, C′ these polars are at once seen to be inclined at an angle of 45° to the line of centres; so the polars of A with respect to X and Y coincide with the polars of A′ with respect to Y and X, thereby proving the first part of the question.

Let P be a point on one of these polars, AB its polar with respect to X (B being taken on the other polar of A). Thon BA′ is the polar of P with respect to Y (on account of the harmonic pencils at A, A′), and B is the pole of PA with respect to the same circle. Let OO′ be the two intersections of opposite lines of the quadrilateral *abcd* which lie on the line BA. These are conjugate points with respect to X; and PO, PO′ are tangents to Y; for, AB are opposite vertices of a quadrilateral circumscribed to X at *abcd*, therefore O, O′ are the common harmonic conjugates to $\alpha\beta$ and AB; and from the circle Y, the tangents from P are harmonic at once with PL, PA and PA, PB, and accordingly coincide with PO, PO′; therefore the tangents from P to Y are conjugate lines to X.

It is worth remarking that an exactly similar proof applies to the corresponding theorem for orthogonal circles (A, A′ being then centres of similitude), which is otherwise proved in TOWNSEND'S *Modern Geometry*, Vol. I., p. 287. Both of course are only particular cases of the co-variant F of two conics breaking up into straight lines, and the above method is general and proves geometrically for two conics so related that the tangents from any point on these lines are harmonic.

5557. (By W. H. H. HUDSON, M.A.)—Find the shape of a uniform wire such that the moment of inertia of any portion of it bounded by two radii vectores about an axis through the pole perpendicular to its plane, may vary as the angle between them.

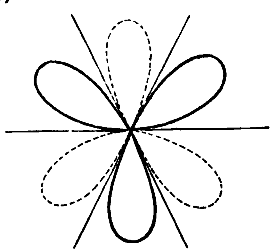
Solution by J. L. MACKENZIE, B.A.; G. S. CARR; *and others.*

Since the moment of inertia increases in the same ratio as the angle θ, we have, by differentiating with respect to θ,

$$r^2\frac{ds}{d\theta} = \text{a constant} = a^3;$$

$$r^2\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} = a^3;\quad \frac{d\theta}{dr} = \frac{r^2}{(a^6-r^6)^{\frac{1}{2}}};$$

and integrating, $\theta = \frac{1}{3}\sin^{-1}\frac{r^3}{a^3}+C.$



If we choose the prime radius so that $C = 0$, we have finally

$$r^3 = a^3\sin 3\theta.$$

The form of the curve is given in the figure. If we give the double sign to the radical in the value of $\frac{d\theta}{dr}$, we get in addition the three dotted loops.

The curve $r^3 = a^3\sin 3\theta$ is the inverse of the cubic $a^3y(3x^2-y^2) = k$, which consists of three simple hyperbolic branches, the asymptotes being evidently three lines through the origin inclined to each other at angles

of 60°. The curve $r^3 = a^3 \sin 3\theta$ becomes (by turning the prime radius through an angle of 30°), $r^3 = a^3 \cos 3\theta$,

one of the important class of curves $r^m = a^m \cos m\theta$. (See SALMON's *Higher Plane Curves*, p. 76, &c.)

5590. (By S. A. RENSHAW.)—If a quadrilateral be inscribed in a conic, show that a point may be found on each of its sides such that, the four points being joined, a quadrilateral inscribed in the former will be formed whose opposite sides produced will meet on the directrix.

Solution by G. MARRO; G. VINCENZO; *and others*.

Questo problema si risolve in un modo più generale, considerando cioé in véce di un quadrangolo inscritto in una conica e della direttrice di questa, un quadrangolo ed una retta qualunque. Siano infatti PQRS e d il quadrangolo e la retta data. Scelgo due punti U, U′ rispettivamente su PQ, PR, indi pel punto M comune alla d e alla UU′, fanno passare tante rette le guali segate con QS, RS determineranno rispettivamente due punteggiate prospettive QS ≡ A, B, C ..., RS ≡ A′, B′, C′ ... Considero ponia i fasci U ≡ A, B, C ..., U′ ≡ A′, B′, C′ ..., essi sono prospettivi per avere l'elunto unito UU′, quindi i raggi corrispondenti UA, U′A′... si segheranno su di una stessa retta u''. Proietto il punto N comune alla u'' e alla d, da U e da U′; e sego UN con QS nel punto T, e la U′N con RS in T′; la TT′ passerá per M, e cosi si sará formato un quadrangolo UU′TT′ inscritto nel dato, e in un due coppie di lati opposti UT, U′T′; UU′, TT′ si segano nei punti N ed M situati nella d che e' quanto si cercava.

Se si fossero costanti i fasci U ≡ A′, B′, C′ ..., U′ ≡ A, B, C ..., e si fosse operato come prendentemente si avrebbe ottenuta un' altra soluzione del problema. Variando poi i punti U, U′ si avramo altrettante soluzioni.

5605. (By J. C. MALET, M.A.)—If a quadric V intersects another quadric U in the planes L and M, and passes through the pole of L with respect to U: prove that it will also pass through the pole of M with respect to U.

I. *Solution by* Professor TOWNSEND, F.R.S.

Denoting by P and Q the two poles in question, by R and S the intersections of the line PQ with L and M, and by X and Y its intersections with U; then, in order to establish the property, it is only necessary to

show that the three pairs of collinear points P and Q, R and S, X and Y are three pairs of conjugates of an involution; which, as XY is cut harmonically by P and R and by Q and S, is manifestly the case; and therefore, &c.

II. *Solution by* E. B. ELLIOTT, M.A.; W. S. F. LONG, B.A.; *and others.*

The equation $V = 0$ must be identical with $U + k\text{LM} = 0$.

Now let (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) be the poles of L and M respectively with regard to U. Then this equation of V may be written

$$U + k\left\{x_1\frac{dU}{dx} + y_1\frac{dU}{dy} + y_1\frac{dU}{dz} + d_1\frac{dU}{dw}\right\}$$
$$\times\left\{x_2\frac{dU}{dx} + y_2\frac{dU}{dy} + z_2\frac{dU}{dz} + w_2\frac{dU}{dw}\right\} = 0.$$

Now (x_1, y, z_1, w_1) satisfies this equation; therefore, remembering EULER's theorem of homogeneous functions, we have

$$U_1 + 2kU_1\left\{x_2\frac{dU}{dx_1} + y_2\frac{dU}{dy_1} + z_2\frac{dU}{dz_1} + w_2\frac{dU}{dw_1}\right\} = 0,$$

that is
$$1 + 2k\left\{x_2\frac{dU}{dx_1} + \ldots\right\} = 0.$$

Therefore, multiplying through by U_2, we have

$$U_2 + k\left\{x_2\frac{dU}{dx_2} + y_2\frac{dU}{dy_2} + \ldots\right\}\left\{x_2\frac{dU}{dx_1} + y_2\frac{dU}{dy_1} + \ldots\right\} = 0;$$

that is to say, (x_2, y_2, z_2, w_2) satisfies $V = 0$.

5497. (By Professor WOLSTENHOLME.)—A heavy uniform chain rests on a smooth arc of a curve in the form of the evolute of a catenary, a length of chain equal to the diameter of curvature at the vertex of the catenary hanging vertically below the cusp; prove (1) that the resolved vertical pressure on the curve per unit is equal to the weight of a unit-length of the chain; (2) that the resolved vertical tension is constant [the chain being, of course, fixed at its highest point]; and (3) that the former property is true for a uniform chain held tightly in contact with the curve whose intrinsic equation is $s = a\sin\phi\,(1 + \cos^2\phi)^{-\frac{3}{2}}$, where ϕ is measured upwards from the horizontal tangent, and the directrix (or straight line from which the tension is measured) is at a depth $a\sqrt{2}$ below the vertex.

I. *Solution by* R. F. DAVIS, M.A.

Measuring s from the cusp and ϕ from the vertical cuspidal tangent, the equation of the curve will be $s = a\tan^2\phi$. If then T be the tension at any point and R the normal resistance at that point per unit-length (weight

w) of chain, $dT = w\,ds \cos\phi$, and $T d\phi + w\,ds \sin\phi = R\,ds$. Hence $\frac{dT}{d\phi} = 2wa \tan\phi \sec\phi$; or $T = 2wa \sec\phi$ (no constant being required for $T_0 = 2wa$). Thus $T\cos\phi = 2wa$, which proves (2). Substituting for T, $R\sin\phi = w$, which proves (1).

To prove (3), measure ϕ from the horizontal tangent at the vertex. Then $dT = w\,ds \sin\phi$; $T d\phi = R\,ds + w\,ds\cos\phi$; and (by the condition given) $R\cos\phi = w$. Thus $T = w\rho \sec\phi\,(1+\cos^2\phi)$, and

$$\rho^{-1}\frac{d\rho}{d\phi} = \tan\phi \cos 2\phi\,(1+\cos^2\phi)^{-1}.$$

Integrating, we have $\rho = \dfrac{ds}{d\phi} = 2a\cos\phi\,(1+\cos^2\phi)^{-\frac{3}{2}}$;

consequently $T = 2wa\,(1+\cos^2\phi)^{-\frac{1}{2}}$, and $T_0 = \sqrt{2}\,.\,wa$. A second integration gives $s = a\sin\phi\,(1+\cos^2\phi)^{-\frac{1}{2}}$.

II. *Solution by the* PROPOSER.

The equations for the first are

$$\frac{dT}{ds} - \kappa g\cos\phi = 0, \qquad T\frac{d\phi}{ds} = R - \kappa g\sin\phi,$$

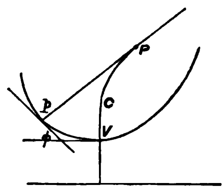


Fig. 1.

ϕ being the angle which the tangent at a point P makes with the vertical, or at the corresponding point p of the catenary with the horizon; also s, the arc VCP, $= a\sec^2\phi$, whence

$$\frac{dT}{d\phi} = \kappa g\,.\,2a\sec\phi\tan\phi, \quad \text{or} \quad T = \kappa g\,.\,2a\sec\phi,$$

no constant being wanted, since $T = \kappa g\,.\,2a$ at the cusp C, where $\phi = 0$, or $T\cos\phi = 2\kappa g a$, that is, the resolved vertical tension is constant.

Again, $$R = \kappa g\sin\phi + T\frac{d\phi}{ds} = \kappa g\left(\sin\phi + \frac{2a\sec\phi}{2a\sec^2\phi\tan\phi}\right)$$
$$= \kappa g\left(\sin\phi + \frac{\cos^2\phi}{\sin\phi}\right) = \frac{\kappa g}{\sin\phi},$$

whence $R\sin\phi = \kappa g$, or the resolved vertical pressure per unit is the weight of a unit-length of the chain.

In general, if y be the height above some fixed horizontal line, we have

$$\frac{dT}{ds} - \kappa g\frac{dy}{ds} = 0, \quad T = \kappa g y,$$

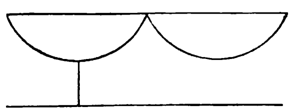


Fig. 2.

no constant being needed if we take the line properly; and

$$T\frac{d\phi}{ds} = R + \kappa g\cos\phi = \frac{\kappa g}{\cos\phi} + \kappa g\cos\phi,$$

if the resolved vertical pressure per unit be κg; whence

$$\frac{dT}{T d\phi} = \frac{\sin\phi\cos\phi}{1+\cos^2\phi}, \quad \text{or} \quad T\,(1+\cos^2\phi)^{\frac{1}{2}} = C \equiv \kappa g c,$$

or $\quad y = \dfrac{c}{(1+\cos^2\phi)^{\frac{1}{2}}}, \quad \dfrac{ds}{d\phi}\sin\phi = \dfrac{c\sin\phi\cos\phi}{(1+\cos^2\phi)^{\frac{3}{2}}}, \quad s = \dfrac{2c\sin\phi}{(1+\cos^2\phi)^{\frac{1}{2}}};$

and when $\phi=0$, $y = \frac{1}{2}c\sqrt{2}$, or the directrix is at a depth $\frac{1}{2}c\sqrt{2}$ below the vertex.

The length of the axis is $\quad c\,(1-\frac{1}{2}\sqrt{2})$,

and the base $\quad = 2c\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{(1+\cos^2\phi)^{\frac{3}{2}}} \equiv c\int_0^{\frac{1}{2}\pi}\frac{\sin^2\phi\,d\phi}{(1+\cos^2\phi)^{\frac{1}{2}}}$,

which is less than $\frac{1}{4}\pi c$ and greater than $\frac{1}{8}\pi c$. Of course its value is at once found from any table of elliptic functions; but the general form of the curve is obvious, being something like a cycloid with the ordinates measured from the axis reduced in a certain ratio.

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139. Note on Professor Monck's Solution of Question 5502. (See pp. 58, 59 of this Volume). *By* W. S. B. Woolhouse, F.R.A.S.

It may not be out of place here to briefly indicate the source of defect in Professor Monck's solution. After Professor Monck has assumed one end, say 5, of the third chord to fall between 1 and 2, I would observe that, as the total positions of 5 must range over the circumference of the circle equally with the other points, it becomes then imperative to consider 1, 5, 2, 3, 4 as five random points in the circumference taken in order; so that, in assigning different positions to the other end, 6, of the third chord, the proper reasoning will be to regard as equal the separate chances that it will lie between 1 and 5, 5 and 2, 2 and 3, 3 and 4, and 4 and 1. This mode of procedure admits a free and equal range to the several points, and leads directly to the true results.

5572. (By Elizabeth Blackwood.)—The centre of a given circle is O, and P and Q are two random points within the circle; find the chance that the triangle POQ is less than a given area.

I. *Solution by* E. B. Seitz.

Let $\text{OP} = x$, $\text{OQ} = y$, r = radius, a = side of square equivalent to given area, $\angle\text{POQ} = \theta$, $\sin^{-1}\left(\dfrac{2a^2}{rx}\right) = \phi$, $\sin^{-1}\left(\dfrac{2a^2}{r^2}\right) = \phi_1$.

Then it is required to find the chance that $xy\sin\theta < 2a^2$. This will be so when $x < \dfrac{2a^2}{r} = x_1$, for all admissible values of θ and y. When $x > x_1$,

the favourable cases occur from $y=0$ to $y=r$ for all values of θ from 0 to ϕ, and for the supplementary values; and from $y=0$ to $y=\dfrac{2a^2}{x\sin\theta}=y_1$, for all values of θ from ϕ to $\frac{1}{2}\pi$, and for the supplementary values. Hence the required chance is

$$p=\frac{2}{\pi^2 r^4}\int_0^{x_1}\int_0^{\pi}\int_0^{r} 2\pi x\,dx\,d\theta\,y\,dy$$

$$+\frac{4}{\pi^2 r^4}\int_{x_1}^{r}\left\{\int_0^{\phi}\int_0^{r} d\theta\,y\,dy+\int_{\phi}^{\pi}\int_0^{y_1} d\theta\,y\,dy\right\}2\pi\,x\,dx$$

$$=\frac{2}{\pi r^2}\int_0^{x_1}\int_0^{\pi} x\,dx\,d\theta+\frac{4}{\pi r^4}\int_{x_1}^{r}\left\{\int_0^{\phi} r^2 x\,d\theta+\int_{\phi}^{\frac{1}{2}\pi}\frac{4a^4}{x}\operatorname{cosec}^2\theta\,d\theta\right\}dx$$

$$=\frac{2}{r^2}\int_0^{x_1} x\,dx+\frac{16a^4}{\pi r^4}\int_{\phi_1}^{\frac{1}{2}\pi}(\phi\cos\phi\operatorname{cosec}^3\phi+\cot^2\phi)\,d\phi$$

$$=\left(\frac{2}{\pi}+\frac{16a^4}{\pi r^4}\right)\sin^{-1}\left(\frac{2a^2}{r^2}\right)-\frac{8a^4}{r^4}+\frac{12a^2(r^4-4a^4)^{\frac{1}{2}}}{\pi r^4}.$$

II. *Solution by the* Proposer.

The chance will be the same if we restrict P and Q to the first quadrant.

Let $OP=x$, $OQ=y$, $POQ=\theta$, $r=$ radius of circle, and $a^2=$ given area.

Employing the notation and principles of Mr. McCol's Calculus of

Table of Limits.			
$y_1=r$	$\theta_1=\dfrac{\pi}{2}$	$x_1=r$	$a_1=\dfrac{r}{\sqrt{2}}$
$y_2=\dfrac{2a^2}{x\sin\theta}$	$\theta_2=\sin^{-1}\left(\dfrac{2a^2}{xr}\right)$	$x_2=\dfrac{2a^2}{r}$	

Equivalent Statements (with which I have recently become acquainted), the required chance is

$$AQ\int d(x^2)\int d\theta\int d(y^2)\div A\int d(x^2)\int d\theta\int d(y^2),$$

in which $A=y_{1'.0}\,\theta_{1'.0}\,x_{1'.0}$, and $Q=p'(\frac{1}{2}xy\sin\theta-a^2)=y_{2'}$.

The denominator of the above fraction is

$$\int_0^{x_1} d(x^2)\int_0^{\theta_1} d\theta\int_0^{y_1} d(y^2)=\frac{\pi r^4}{2},$$

so that the required chance $=AQ\int\frac{8}{\pi r}\,x\,dx\int d\theta\int y\,dy.$

Now $\quad AQ=y_{2'.1'.0}\,\theta_{1'.0}\,x_{1'.0}$, and $y_{2'.1'}=y_{2'}\alpha+y_{1'}\beta$,

in which $\quad \alpha = p'(y_2 - y_1) = p\left(\sin\theta - \dfrac{2a^2}{xr}\right) = \theta_2 p\,(xr - 2a^2) = \theta_2 x_2,$

and $\quad \beta = p'(y_1 - y_2) = p'\left(\sin\theta - \dfrac{2a^2}{xr}\right) = \theta_{2'} x_2 + x_{2'}.$

Substituting, we get

$$\mathrm{AQ} = y_{2'.0}\theta_{1'.2.\underset{\cdot}{0}}x_{1'.2.\underset{\cdot}{0}} + y_{1'.0}\theta_{2'.\underset{\cdot}{1'}.0}x_{1'.2.\underset{\cdot}{0}} + y_{1'.0}\theta_{1'.0}x_{2'.1'.0}.$$

The factors dotted underneath may be cancelled, as they are evidently implied in their co-factors of the same variable; also

$$x_{2'.1'} = x_{2'}a_{1'} + x_{1'}a_1, \quad \text{and} \quad x_{1'.2} = x_{1'.2}a_{1'}.$$

Hence, finally,

$$\mathrm{AQ} = a_{1'}(y_{2'.0}\theta_{1'.2}x_{1'.2} + y_{1'.0}\theta_{2'.0}x_{1'.2} + y_{1'.0}\theta_{1'.0}x_{2'.0}) + a_1(y_{1'.0}\theta_{1'.0}x_{1'.0}).$$

The last term gives the chance for the statement a_1; that is to say, for the case in which $a > \dfrac{r}{\sqrt{2}}$; and this chance is

$$\frac{8}{4\pi r^4}\int_0^{x_1} x\,dx \int_0^{\theta_1} d\theta \int_0^{y_1} y\,dy = 1.$$

The other three terms give the chance for the statement $a_{1'}$; that is, when $a < \tfrac{1}{2}r\sqrt{2}$, and this chance is

$$\frac{8}{\pi r^4}\left\{\int_{x_2}^{x_1} x\,dx \int_{\theta_2}^{\theta_1} d\theta \int_0^{y_2} y\,dy + \int_{x_2}^{x_1} x\,dx \int_0^{\theta_2} d\theta \int_0^{y_1} y\,dy + \int_0^{x_2} x\,dx \int_0^{\theta_1} d\theta \int_0^{y_1} y\,dy\right\}.$$

[This is identical with Mr. Seitz's expression for the required chance.]

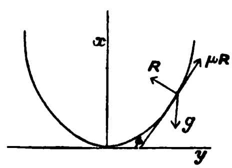
5606. (By R. E. Riley, B.A.)—A particle slides down a rough parabola whose axis is vertical, starting from an extremity of the latus-rectum. If it stops at the vertex, prove that $\mu = \pi^{-1}\log_e 4$.

Solution by J. R. White, B.A.; R. F. Davis, B.A.; *and others.*

The equations to the parabola are

$$y = 2am, \qquad x = am^2,$$

where $m = -\tan\phi$; and the equations of motion along the tangent and normal are



$$\frac{1}{2}\cdot\frac{d(v^2)}{ds} = \mu\mathrm{R} - g\sin\phi,$$

$$\frac{v^2}{\rho} = \mathrm{R} - g\cos\phi.$$

Eliminating R, we get

$$\frac{dz}{d\phi} - 2\mu z = 2g\,\frac{ds}{d\phi}\,(\mu\cos\phi - \sin\phi$$

$$(z \equiv v^2) \qquad\qquad = 4ga\,(\mu\sec^2\phi - \sec^2\phi\,.\,\tan\phi).$$

Multiply across by $\epsilon^{-2\mu\phi}$, and integrate; then

$$z\,\epsilon^{-2\mu\phi} = C - 2ga\sec^2\phi\,.\,\epsilon^{-2\mu\phi}.$$

Initially, $0 = C - ga\,.\,\epsilon^{-\frac{1}{2}(\mu\pi)}$, and finally, $0 = C - 2ga\,.\,\epsilon^{-2\mu\pi}$;

therefore $\qquad 2\epsilon^{-\frac{1}{2}(\mu\pi)} = 1,\quad$ whence $\mu = \dfrac{\log_e 4}{\pi}$.

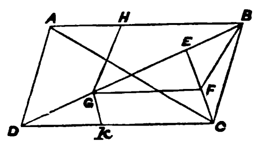
[In the Solution of Quest. 5103 (*Reprint*, Vol. XXVII., p. 50), an investigation is given of the motion of a particle down a rough parabolic arc whose axis is vertical and concavity upwards; wherein it is shown that, if ϕ be measured from the horizontal tangent at the vertex, we have

$$v^2 = 2ag\,(C\,e^{2\mu\phi} - \sec^2\phi)\,;$$

and, by the conditions of the question, the velocity vanishes both when $\phi = 0$ and $\phi = \frac{1}{4}\pi$; thus, $e^{\frac{1}{2}(\mu\pi)} = 2$, &c.]

5623. (By C. K. Pillai.)—ABCD is a parallelogram; a point E is taken in the diagonal BD, and a point F in CE; also FG is drawn parallel to DC meeting the diagonal in G, and GH is drawn from G parallel to BF, and meeting AB in H; prove that AH : AB = DG : DE.

Solution by D. J. McAdam; J. O'Regan; W. S. F. Long; *and others.*



Draw CK parallel to CE; then

$$CK = FG = BH,$$

therefore $\qquad DK = AH$;

hence we have

$$AH : AB = DK : DC = DG : DE.$$

5544. (By J. J. Walker, M.A.)—Writing Hermite's series,

$$F(x) = \frac{1}{n} + \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} + \frac{x^3}{n(n+1)(n+2)(n+3)} + \ldots,$$

if $$f(x) = \frac{1}{n} - \frac{x}{(n+1)\,.\,1} + \frac{x^2}{(n+2)\,1\,.\,2} - \frac{x^3}{(n+3)\,1\,.\,2\,.\,3} + \ldots,$$

prove that $\dfrac{F(x)}{f(x)} = e^x$. [See Question 5492, pp. 76—78 of this Volume.]

I. *Solution by* J. O. JELLY, M.A.; E. RUTTER; R. RAWSON; *and others.*

$$e^{-x}F(x)=\left\{\frac{1}{n}+\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}+\frac{x^3}{n(n+1)(n+2)(n+3)}+\dots\right\}$$
$$\times\left\{1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\dots\right\}$$
$$=\frac{1}{n}-\frac{x}{(n+1)1}+\frac{x^2}{(n+2)1.2}-\frac{x^3}{(n+3)1.2.3}+\dots=f(x).$$

II. *Solution by* J. HAMMOND, M.A.; E. RUTTER; *and others.*

Let $\dfrac{x^n F(x)}{\Gamma(n)}=u=\dfrac{x^n}{\Gamma(x+1)}+\dfrac{x^{n+1}}{\Gamma(n+2)}+\dots$; then $\dfrac{du}{dx}=\dfrac{x^{n-1}}{\Gamma(n)}+u.$

Solving this equation in the usual manner, we obtain

$$u=\frac{e^x}{\Gamma(n)}\int_0^x e^{-x}x^{n-1}dx=\frac{e^x x^n}{\Gamma(n)}\int_0^1 e^{-xy}y^{n-1}dy;$$

hence $$F(x)=e^x\int_0^1 e^{-xy}y^{n-1}dy$$
$$=e^x\int_0^1\left(y^{n-1}-\frac{x}{1}y^n+\frac{x^2}{1.2}y^{n+1}-\frac{x^3}{1.2.3}y^{n+2}+\text{\&c.}\right)dy$$
$$=e^x\left\{\frac{1}{n}-\frac{x}{(n+1).1}+\frac{x^2}{(n+2).1.2}-\frac{x^3}{(n+3).1.2.3}+\text{\&c.}\right\}.$$

If the series in brackets be written $f(x)$, we obtain $\dfrac{F(x)}{f(x)}=e^x$.

The truth of this result is evident in the particular case $n=1$, when $F(x)$ and $f(x)$ become respectively

$$F(x)=1+\frac{x}{1.2}+\frac{x^2}{1.2.3}+\dots=\frac{e^x-1}{x},$$
$$f(x)=1-\frac{x}{1.2}+\frac{x^2}{1.2.3}-\dots=\frac{1-e^{-x}}{x}.$$

III. *Solution by the* PROPOSER.

The first solution is only a verification by actual multiplication to a limited number of terms. The proof may be completed as follows:—

In the product of $e^{-x}F(x)$, the coefficient of x^r is easily seen to be

$$\frac{1}{n.n+1\dots n+r}\left(1-\frac{n+r}{1}+\frac{n+r.n+r-1}{1.2}-\dots+\dots\text{ to }(r+1)\text{ terms}\right),$$

in which the terms within brackets make up the coefficient of $(-x)^r$ in $(1+x)^{n+r}(1-x+x^2-x^3\dots)$, or in $(1+x)^{n+r-1}$; viz., $\dfrac{n+r-1.n+r-2\dots n}{1.2\dots r}$.

The coefficient of x^r in the product $e^{-x}F(x)$ is therefore $\dfrac{(-1)^r}{1.2\dots r(n+r)}$, and the product in question is thus identified with $f(x)$.

I was led to observe this identity from the differential equation $x\mathrm{F}'(x)+(n-x)\,\mathrm{F}(x)=n$ in my solution of M. HERMITE's Question 5492 from which

$$x^n e^{-x}\,\mathrm{F}'(x)+(nx^{n-1}-x^n)\,e^{-x}\,\mathrm{F}(x)=nx^{n-1}e^{-x},$$

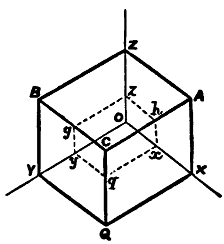
and integrating $\quad x^n e^{-x}\,\mathrm{F}(x)=x^n-\dfrac{n}{n+1}\,x^{n+1}+\dfrac{n}{n+2}\cdot\dfrac{x^{n+2}}{1\,.\,2}-\dots,$

or $\quad \dfrac{\mathrm{F}(x)}{n}=e^x\left(\dfrac{1}{n}-\dfrac{x}{n+1}+\dfrac{x^2}{n+2\,.\,1\,.\,2}-\dots\right),$

where $\dfrac{\mathrm{F}(x)}{n}$ is the $\mathrm{F}(x)$ of Question 5544.

5315. (By Colonel A. R. CLARKE, C.B., F.R.S.)—A straight line intersects a cube; show that the chance that the intercepted segment is less than the side of the cube is $\dfrac{13}{6\pi}$.

Solution by the PROPOSER.



Let AXQYBZ be the orthographical projection of a cube. Let a, α; b, β; c, γ be the sines and cosines of the angles the sides OX, OY, OZ make with the normal to the plane of projection. Then, if the side of the cube be unity, the projected lengths of the sides OX, OY, OZ will be a, b, c, and the projected areas of the faces ACQX, BCQY, ACBZ will be α, β, γ respectively. Let a line perpendicular to the plane of projection meet the cube; it will be represented by a point, as s. If the point t represent another such line, and if s and t be in a line parallel to the edge QY of the cube, and are not separated by any line of the figure, then the segments of s and t intercepted by the cube are clearly equal. Trace the dotted lines qx, xh, hz, zg, gy, yq parallel respectively to QX, XA, AZ, ZB, BY, YQ; this dotted line (forming a hexagon) is the locus of lines from which equal segments are intercepted by the cube. Points *inside* this hexagon correspond to intercepted segments greater, and points outside it to segments less, than those which correspond to points on the perimeter of the dotted hexagon. To determine the length of the intercepted segment of lines corresponding to this dotted line, take the intersecting line that corresponds to q on the edge CQ. If θ be the inclination of this edge to the normal to the plane of projection, it is easy to see that the intercepted segment is $\mathrm{Q}q\,(\tan\theta+\cot\theta)=\dfrac{\mathrm{Q}q}{c\gamma}$. If then we take the dotted hexagon to separate intercepted segments greater than unity from those less than unity, we must make $\mathrm{Q}q=c\gamma$; therefore $\mathrm{C}q=c(1-\gamma)$. Similarly, $\mathrm{C}h=b(1-\beta)$ and $\mathrm{C}g=a(1-\alpha)$. The hexagon

is formed by three parallelograms whose areas are

$$Chqx = \alpha\left(\frac{Ch}{CA}\right)\frac{Cq}{CQ} = \alpha\,(1-\beta)\,(1-\gamma),$$

$$Cgyq = \beta\left(\frac{Cg}{CB}\right)\frac{Cq}{CQ} = \beta\,(1-\alpha)\,(1-\gamma),$$

$$Chzg = \gamma\left(\frac{Ch}{CA}\right)\frac{Cg}{CB} = \gamma\,(1-\beta)\,(1-\alpha).$$

The area of the projection of the cube, viz., $\alpha+\beta+\gamma$, representing then the *total* number of lines intersecting (perpendicularly to the plane of projection) the cube, the number of lines which have intercepted segments shorter than the side of the cube is $2\alpha\beta+2\beta\gamma+2\gamma\alpha-3\alpha\beta\gamma$.

Produce the sides OX, OY, OZ of the cube to meet in X′, Y′, Z′ a sphere of infinite radius and centre O. Take any point P in the quadrantal spherical triangle X′Y′Z′; then, if OP correspond to the normal to the plane of projection we have been considering, we have $\alpha = \cos PX'$, $\beta = \cos PY'$, $\gamma = \cos PZ'$. We must suppose that from every point P in this spherical triangle a number $=\alpha+\beta+\gamma$ (varying with P) of parallel lines are drawn to intersect the cube. Hence, if $d\sigma$ be the element of surface at P, the total number of lines intersecting the cube is

$$N = \int(\alpha+\beta+\gamma)\,d\sigma,$$

whilst of these the number of favourable cases is

$$n = \int(2\alpha\beta+2\beta\gamma+2\alpha\gamma-3\alpha\beta\gamma)\,d\sigma,$$

the integration in both cases extending over the surface X′Y′Z′. But as the expressions for N, n are both symmetrical with respect to $\alpha\beta\gamma$, we may substitute for them

$$n = \int(2\alpha\gamma-\alpha\beta\gamma)\,d\sigma, \quad\text{and}\quad N = \int\gamma\,d\sigma.$$

Now $\gamma\,d\sigma$ is the element of surface projected on to the coordinate plane OX′Y′; hence $N = \frac{1}{4}\pi$. And if we adopt rectangular coordinates,

$$n = \int_0^1\int_0^{\sqrt{1-y^2}}(2x-xy)\,dy\,dx = \tfrac{13}{24}.$$

Therefore the required probability is $\dfrac{13}{6\pi}$.

[This question is analogous to a very elegant one solved by Mr. Woolhouse on pp. 87–92 of the *Lady's and Gentleman's Diary* for 1860.]

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